École polytechnique fédérale de Lausanne

Master project

Master in Mathematics

Categorical probability

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1 Introduction

Probability theory has been developed by measure theory. We want to see to what extend does this theory can be generalised to theory of categories. This could bring a more conceptual understanding of probabilities, and we could generalise some results from statistical theory. This is interesting because this could bring uniform results applying to different model of probabilities, also those results could apply in other contexts and lastly we could find new results thanks to categorical theory. The idea of categorical probability is to understand random spaces as objects of a category, and probability densities as morphisms in this category. From here, a great interest has been in how monads can construct Markov categories, for example the Giry monad constructed by Michèle Giry in *A categorical approach to probability theory* [3], and on how to theorize information theory, for example the work of Peter Golubtsov in *Method of Additional Structures on the Objects of a Monoidal Kleisli Category as a Background for Information Transformers Theory* [4]. In this project, we will essentially follow the work of Tobias Fritz in *A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics* [2]. In Section 2, we state the context, define the notion of Markov categories as a generalisation of probability spaces and densities, and give some examples coming from the classical statistical theory. Then in Section 3, we attempt to explain and develop some of the properties we can add on an arbitrary Markov category, and what it implies. Lastly, in Section 4, we explain some generalisations of characteristics and theorems from statistical theory.

2 Markov Categories

In this section, we will construct categories in which morphisms can be interpreted as conditional probability densities. We will see some examples of these, and a way to construct new ones from old.

2.1 Definitions

First it is important to define some important objects, that we will use in order to construct the categories we are interested in. Those categories will need to be *symmetric monoidal*, defined as follows.

Definition 2.1. [1, 6.1.1.] A *monoidal category* $\mathcal{M}$ consists of:

1. a category $\mathcal{M}$;
2. a bifunctor
$$\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M},$$
called the *tensor product*, where $x \otimes y$ denotes the image under $\otimes$ of the pair (x, y);
3. an object $I \in \mathcal{M}$ called the *unit object*;
4. for every triple x, y, z of objects, a natural isomorphism
$$a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z),$$
called the *associator*;
5. for every object x, a natural isomorphism
$$\lambda_x : I \otimes x \to x,$$
called the *left unitor*;
6. for every object x, a natural isomorphism
$$\rho_x : x \otimes I \to x,$$
called the *right unitor*.

We then require that the associator obey to the *pentagon identity*, which says that for all objects w, x, y, z in $\mathcal{M}$, this diagram commutes:

$$\begin{tikzcd}
 & & (w \otimes x) \otimes (y \otimes z) \arrow[drr, "a_{w,x,y\otimes z}", "\cong"'] & & \\
((w \otimes x) \otimes y) \otimes z \arrow[urr, "a_{w\otimes x,y,z}", "\cong"'] \arrow[dr, "\cong", "a_{w,x,y}\otimes 1_z"'] & & & & w \otimes (x \otimes (y \otimes z)). \\
 & (w \otimes (x \otimes y)) \otimes z \arrow[rr, "\cong", "a_{w,x\otimes y,z}"'] & & w \otimes ((x \otimes y) \otimes z) \arrow[ur, "\cong", "1_w \otimes a_{x,y,z}"'] &
\end{tikzcd}$$

We demand that the unitors obey to the *triangle identity*, which says that for all objects x, y in $\mathcal{M}$, this diagram commutes:

$$\begin{tikzcd}
(x \otimes I) \otimes y \arrow[rr, "a_{x,I,y}", "\cong"'] \arrow[dr, "\cong", "\rho_x \otimes 1_y"'] & & x \otimes (I \otimes y). \arrow[dl, "\cong"', "1_x \otimes \lambda_y"] \\
 & x \otimes y &
\end{tikzcd}$$

Definition 2.2. With the notation of 2.1, a monoidal category is *symmetric* when, moreover, for every x, y in $\mathcal{M}$ there is a natural isomorphism

$$B_{x,y} : x \otimes y \to y \otimes x,$$

called the *braiding*. We require that:

- the braiding and associator obey to the first *hexagon identity*, which says that for every triple x, y, z of objects, this diagram commutes:

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{x,y,z}", "\cong"'] \arrow[d, "B_{x,y}\otimes 1_z"', "\cong"] & x \otimes (y \otimes z) \arrow[r, "B_{x,y\otimes z}", "\cong"'] & (y \otimes z) \otimes x \arrow[d, "\cong"', "a_{y,z,x}"] \\
(y \otimes x) \otimes z \arrow[r, "\cong", "a_{y,x,z}"'] & y \otimes (x \otimes z) \arrow[r, "\cong", "1_y \otimes B_{x,z}"'] & y \otimes (z \otimes x).
\end{tikzcd}$$

- the *unit coherence*, which says that for every object x, this diagram commutes:

$$\begin{tikzcd}
x \otimes 1 \arrow[r, "B_{x,1}", "\cong"'] \arrow[dr, "\cong", "\rho_x"'] & 1 \otimes x \arrow[d, "\cong"', "\lambda_x"] \\
 & x,
\end{tikzcd}$$

- the *symmetry axiom*, which says that for every pair of objects x, y, this diagram commutes:

$$\begin{tikzcd}
x \otimes y \arrow[r, "B_{x,y}"] \arrow[dr, equal] & y \otimes x \arrow[d, "B_{y,x}"] \\
 & x \otimes y,
\end{tikzcd}$$

Remark. The symmetry axiom in 2.2 and the first hexagon identity together imply the second hexagon identity, i.e. for every triple of objects x, y, z, this diagram commutes:

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "a^{-1}_{x,y,z}"] \arrow[d, "1_x \otimes B_{y,z}"'] & (x \otimes y) \otimes z \arrow[r, "B_{x\otimes y,z}"] & z \otimes (x \otimes y) \arrow[d, "a^{-1}_{z,x,y}"] \\
x \otimes (z \otimes y) \arrow[r, "a^{-1}_{x,z,y}"'] & (x \otimes z) \otimes y \arrow[r, "B_{x,z}\otimes 1_y"'] & (z \otimes x) \otimes y,
\end{tikzcd}$$

The following coherence theorem will simplify some calculations in symmetric monoidal categories for the next section.

Theorem 2.3. *[5, 7.2.1.] Let C be a symmetric monoidal category. Every diagram in C made up of associators, unitors and braidings, and in which both sides have the same underlying permutation, commutes.*

We are now interested in how to define functors between monoidal categories that preserve the structure. These functors are called *lax monoidal functors*.

Definition 2.4. Let C and $\mathcal{D}$ be monoidal categories with monoidal structures $\otimes_C$ and $\otimes_{\mathcal{D}}$, respectively. A functor $F : C \to \mathcal{D}$, together with a natural transformation $\nabla_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_C Y)$ and a morphism $\eta : I_{\mathcal{D}} \to F(I_C)$, called the unit, is a *lax monoidal functor* if it respects the monoidal structure, i.e., if the following diagrams commute for all objects X, Y, Z in C:

$$\begin{tikzcd}
(F(X) \otimes_{\mathcal{D}} F(Y)) \otimes_{\mathcal{D}} F(Z) \arrow[r, "\alpha_{\mathcal{D}}"] \arrow[d, "\nabla_{X,Y} \otimes_{\mathcal{D}} 1_{F(Z)}"'] & F(X) \otimes_{\mathcal{D}} (F(Y) \otimes_{\mathcal{D}} F(Z)) \arrow[d, "1_{F(X)} \otimes_{\mathcal{D}} \nabla_{Y,Z}"] \\
F(X \otimes_C Y) \otimes_{\mathcal{D}} F(Z) \arrow[d, "\nabla_{X \otimes_C Y, Z}"'] & F(X) \otimes_{\mathcal{D}} F(Y \otimes_C Z) \arrow[d, "\nabla_{X, Y \otimes_C Z}"] \\
F((X \otimes_C Y) \otimes_C Z) \arrow[r, "F(\alpha_C)"'] & F(X \otimes_C (Y \otimes_C Z)),
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
I_{\mathcal{D}} \otimes_{\mathcal{D}} F(X) \arrow[r, "\eta \otimes_{\mathcal{D}} 1_{F(X)}"] \arrow[d, "\lambda_{\mathcal{D}}"'] & F(I_C) \otimes_{\mathcal{D}} F(X) \arrow[d, "\nabla_{I_C, X}"] \\
F(X) & F(I_C \otimes_C X), \arrow[l, "F(\lambda_C)"]
\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}
F(X) \otimes_{\mathcal{D}} I_{\mathcal{D}} \arrow[r, "1_{F(X)} \otimes_{\mathcal{D}} \eta"] \arrow[d, "\rho_{\mathcal{D}}"'] & F(X) \otimes_{\mathcal{D}} F(I_{\mathcal{D}}) \arrow[d, "\nabla_{X, I_C}"] \\
F(X) & F(X \otimes_C I_C), \arrow[l, "F(\rho_C)"]
\end{tikzcd} \tag{3}$$

where we omitted the indices in α, λ and ρ for simplicity.

If, in addition, a functor preserves the symmetric structure, then it is called a *lax symmetric monoidal functor*.

Definition 2.5. Let C and $\mathcal{D}$ be symmetric monoidal categories. A lax monoidal functor $F : C \to \mathcal{D}$ is *symmetric* if, for all objects X, Y in C, the diagram

$$\begin{tikzcd}
F(X) \otimes_{\mathcal{D}} F(Y) \arrow[r, "B_{\mathcal{D}}"] \arrow[d, "\nabla_{X,Y}"'] & F(Y) \otimes_{\mathcal{D}} F(X) \arrow[d, "\nabla_{Y,X}"] \\
F(X \otimes_C Y) \arrow[r, "F(B_C)"'] & F(Y \otimes_C X),
\end{tikzcd} \tag{4}$$

commutes.

If the structure of the functor is made of isomorphisms, then it is called a *strong symmetric monoidal functor*.

Definition 2.6. Let C and $\mathcal{D}$ be monoidal categories. A lax (symmetric) monoidal functor $F : C \to \mathcal{D}$ is *strong* (*strict*) if its coherence maps $\nabla_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_C Y)$ and $\eta : I_{\mathcal{D}} \to F(I_C)$ are isomorphisms (identities).

Some objects, called *monads on a category* will be useful to construct new categories from old, having the probabilistic properties we want.

Definition 2.7. [1, 4.1.1.] A *monad* on a category C is a triple (T, η, μ) where $T : C \to C$ is a functor and $\eta : 1_C \implies T, \mu : TT \implies T$ are natural transformations satisfying the commutativity conditions

$$\mu \circ (\eta * 1_T) = 1_T = \mu \circ (1_T * \eta), \quad \mu \circ (\mu * 1_T) = \mu \circ (1_T * \mu),$$

i.e., the diagrams

$$\begin{tikzcd}
T \arrow[r, "\eta * 1_T"] \arrow[dr, equal] & TT \arrow[d, "\mu"] & T, \arrow[l, "1_T * \eta"'] \arrow[dl, equal] \\
& T &
\end{tikzcd} \tag{5}$$

$$\begin{array}{ccc} TTT & \xrightarrow{\mu * 1_T} & TT \\ {\scriptstyle 1_T * \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow[\mu]{} & T, \end{array} \tag{6}$$

commute.

Definition 2.8. Let $\mathcal{D}$ be a symmetric monoidal category. Further, let $T : \mathcal{D} \to \mathcal{D}$ be a monad with multiplication $\mu : TT \to T$ and unit $\eta : I \to T$. It is a *symmetric monoidal monad* if it comes equipped with morphisms

$$\nabla_{X,Y} : TX \otimes TY \to T(X \otimes Y),$$

natural in X and Y, which makes T into a lax symmetric monoidal functor with unit $\eta_I : I_{\mathcal{D}} \to TI_{\mathcal{D}}$, and such that μ and η are monoidal transformations. Concretely, the latter means that the diagrams

$$\begin{array}{ccccc} TTX \otimes TTY & \xrightarrow{\nabla_{TX,TY}} & T(TX \otimes TY) & \xrightarrow{T\nabla_{X,Y}} & TT(X \otimes Y) \\ {\scriptstyle \mu_X \otimes \mu_Y}\downarrow & & & & \downarrow{\scriptstyle \mu_{X \otimes Y}} \\ TX \otimes TY & & \xrightarrow[\nabla_{X,Y}]{} & & T(X \otimes Y), \end{array} \tag{7}$$

$$\begin{array}{ccc} & X \otimes Y & \\ {\scriptstyle \eta_X \otimes \eta_Y}\swarrow & & \searrow{\scriptstyle \eta_{X \otimes Y}} \\ TX \otimes TY & \xrightarrow[\nabla_{X,Y}]{} & T(X \otimes Y), \end{array} \tag{8}$$

must commute for all $X, Y \in \mathcal{D}$.

Definition 2.9. Let $\mathcal{D}$ be a symmetric monoidal category with unit I being terminal. A monad T on $\mathcal{D}$ is *affine* if $TI \cong I$. By this we mean that the unique morphism $TI \to I$ has inverse $\eta_I : I \to TI$.

We can use the monads to construct new categories from old. We will now define the constructions of the *Eilenberg-Moore* and *Kleisli* categories.

Definition 2.10. [1, 4.1.2] Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category C. By an *algebra on this monad* is meant a pair (C, ξ) where $C \in C$, $\xi : T(C) \to C$ and $\xi \circ \eta_C = 1_C$, $\xi \circ T(\xi) = \xi \circ \mu_C$, i.e., the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & T(C) \\ & \searrow{=} & \downarrow{\scriptstyle \xi} \\ & & C \end{array} , \qquad \begin{array}{ccc} TT(C) & \xrightarrow{\mu_C} & T(C) \\ {\scriptstyle T(\xi)}\downarrow & & \downarrow{\scriptstyle \xi} \\ T(C) & \xrightarrow[\xi]{} & C, \end{array}$$

commute. If (D, ξ) is another $\mathbb{T}$-algebra, a morphism $f : (C, \xi) \to (D, \zeta)$ of $\mathbb{T}$-algebras is a morphism $f : C \to D$ of C such that $f \circ \xi = \zeta \circ T(f)$, i.e., the diagram

$$\begin{array}{ccc} T(C) & \xrightarrow{T(f)} & T(D) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \zeta} \\ C & \xrightarrow[f]{} & D, \end{array}$$

commutes.

As said above, we can construct new categories from old thanks to the monads.

Proposition 2.11. *[1, 4.1.3] Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category C. The $\mathbb{T}$-algebras and their morphisms constitute a category, written $C^{\mathbb{T}}$ and called the* Eilenberg-Moore algebra *of the monad and*

- *for all morphisms of $\mathbb{T}$-algebras $f : (C, \xi) \to (D, \zeta)$, $g : (D, \zeta) \to (E, \chi)$ the* composition *$g \circ f : (C, \xi) \to (D, \zeta) \to (E, \chi)$ is defined by $g \circ f$ in C, so that the diagram*

$$\begin{tikzcd} T(C) \arrow[r, "T(f)"] \arrow[rr, bend left, "T(g \circ f)"] \arrow[d, "\xi"'] & T(D) \arrow[r, "T(g)"] \arrow[d, "\zeta"] & T(E) \arrow[d, "\chi"] \\ C \arrow[r, "f"'] & D \arrow[r, "g"'] & E, \end{tikzcd}$$

  commutes;

- *for all* $\mathbb{T}$*-algebras* (C,ξ)*, the* identity *of* $\mathbb{T}$*-algebras* $1_C : (C,\xi) \to (C,\xi)$ *is the identity on* C $1_C : C \to C$*, so that the diagram*

$$\begin{tikzcd} T(C) \arrow[r, equal, "T(1_C)"] \arrow[d, "\xi"'] & T(C) \arrow[d, "\xi"] \\ C \arrow[r, equal] & C, \end{tikzcd}$$

commutes.

Proposition 2.12. *[1, 4.1.4]. Let* $\mathbb{T} = (T,\eta,\mu)$ *be a monad on a category* C*. Consider the forgetful functor*

$$U : C^{\mathbb{T}} \longrightarrow C, \quad (C,\xi) \mapsto C, \quad f \mapsto f.$$

1. *U is faithful;*
2. *U reflects isomorphisms i.e. if* $f : (C,\xi) \to (D,\zeta)$ *is a morphism in* $C^{\mathbb{T}}$ *and* $U(f)$ *is an isomorphism in* C*, then* f *is an isomorphism;*
3. *U has a left adjoint.*

The left adjoint F *of* U *maps* $C \in C$ *to* $(T(C),\mu_C)$ *and* $f : C \to C'$ *to* $T(f) : (T(C),\mu_C) \to (T(C'),\mu_{C'})$*. The unit of the adjunction is just* $\eta : 1_C \Rightarrow UF = T$ *and the counit* $\epsilon : FU \to 1_{C^{\mathbb{T}}}$ *is given by* $\epsilon_{(C,\xi)} = \xi$*.*

Definition 2.13. [1, 4.1.5]. Let $\mathbb{T} = (T,\eta,\mu)$ be a monad on a category C. Consider the forgetful functor $U = C^{\mathbb{T}} \to C$ and its left adjoint $F : C \to C^{\mathbb{T}}$ mapping C to $(T(C),\eta_C)$. A $\mathbb{T}$-algebra is *free* when it is isomorphic to one of the form $F(C) = (T(C),\mu_C)$, for some object $C \in C$.

Proposition 2.14. *[1, 4.1.6]. Let* $\mathbb{T} = (T,\eta,\mu)$ *be a monad on a category* C*. The full subcategory of* $C^{\mathbb{T}}$ *generated by the free* $\mathbb{T}$*-algebras is equivalent to the following category* $Kl(\mathbb{T})$*.*

- *The objects of* $Kl(\mathbb{T})$ *are those of* C*;*
- *A morphism* $f : C \to D$ *in* $Kl(\mathbb{T})$ *is a morphism* $\underline{f} : C \to T(D)$ *in* C*;*
- *The composite of two morphisms* $f : A \to B$*,* $g : B \to C$ *in* $Kl(\mathbb{T})$ *is given in* C *by the composite*

$$A \xrightarrow{\underline{f}} T(B) \xrightarrow{T(\underline{g})} TT(C) \xrightarrow{\mu_C} T(C);$$

- *The identity on an object* C *of* $Kl(\mathbb{T})$ *is just* $\eta_C : C \to T(C)$ *in* C*.*

The category $Kl(\mathbb{T})$ *is called the* Kleisli *category of the monad* $\mathbb{T}$*. The standard inclusion functor* $F : C \to C_{\mathbb{T}}$ *acts as the identity on objects, and for all morphism* $f : C \to D$ *in* C*, we define* $F(f) : C \to D$ *in* $Kl(\mathbb{T})$ *by its representation* $\underline{F(f)} : C \to T(D)$ *in* C *given by the composition,*

$$C \xrightarrow{f} D \xrightarrow{\eta_D} T(D).$$

Let us stop here for a moment, and give an example of monad to illustrate.

Example 2.15. Let *FinSet* be the category of sets and fonctions between them. Now let $T : FinSet \to FinSet$ be the functor defined as follows.

For all objects X in *FinSet*,

$$TX := \mathcal{P}(X)\backslash\{0\},$$

where $\mathcal{P}(X)$ is the power set of X.

For all morphisms $f : X \to Y$ in *FinSet*, for all nonempty subsets $\mathcal{U}$ of X,

$$Tf(\mathcal{U}) := \bigcup_{x\in\mathcal{U}} f(x) \subseteq Y.$$

It is not empty since $\mathcal{U}$ is not, so it is in TY.

We show that $T : FinSet \to FinSet$ is a functor. For all objects X in C, for all $\mathcal{U}$ in X,

$$T(1_X)(\mathcal{U}) = \mathcal{U} = 1_{TX}(\mathcal{U}),$$

and for all morphisms $f : X \to Y$, $g : Y \to Z$, for all $\mathcal{U}$ in X,

$$T(g \circ f)(\mathcal{U}) = \bigcup_{x \in \mathcal{U}} (g \circ f)(x) = \bigcup_{y \in f(\mathcal{U})} g(y) = \bigcup_{y \in Tf(\mathcal{U})} g(y) = Tf \circ Tg(\mathcal{U}),$$

and T is a functor.

For T to be a monad, we need to define the natural transformations $\eta : 1_{Set} \Rightarrow T$ and $\mu : TT \Rightarrow T$. The transformation η is defined as follows. For all objects X in *FinSet*, and for all $x \in X$,

$$\eta_X(x) = \{x\}.$$

This defines a natural transformation since for all morphisms $f : X \to Y$ in *FinSet*, for all x in X,

$$Tf \circ \eta_X(x) = Tf(\{x\}) = \{f(x)\} = \eta_Y(f(x)) = \eta_Y \circ f(x),$$

and the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & TX \arrow[d, "T(f)"] \\ Y \arrow[r, "\eta_Y"'] & TY. \end{tikzcd}$$

The transformation μ is defined as follows. For all objects X in *FinSet*, and for all nonempty subsets S of $\mathcal{P}(X)\backslash\{\emptyset\}$, i.e., $S \in \mathcal{P}(\mathcal{P}(X)\backslash\{\emptyset\})\backslash\{\emptyset\} = TTX$, $\mu_X(S)$ is the smallest object of $TX = \mathcal{P}(X)\backslash\{\emptyset\}$ containing all the sets in S, i.e.,

$$\mu_X(S) := \bigcup_{\mathcal{U} \in S} \mathcal{U}.$$

This defines a natural transformation since for all morphisms $f : X \to Y$ in *FinSet*, for all S in TTX,

$$Tf \circ \mu_X(S) = Tf\left(\bigcup_{\mathcal{U} \in S} \mathcal{U}\right) = \bigcup_{\mathcal{U} \in S} \bigcup_{x \in \mathcal{U}} f(x) = \bigcup_{\mathcal{U} \in S} Tf(\mathcal{U}) = \mu_{TX}\left(\bigcup_{\mathcal{U} \in S} Tf(\mathcal{U})\right) = \mu_{TX} \circ TTf(S),$$

and the following diagram commutes

$$\begin{tikzcd} TTX \arrow[r, "\eta_X"] \arrow[d, "TT(f)"'] & TX \arrow[d, "T(f)"] \\ TTY \arrow[r, "\mu_Y"'] & TY. \end{tikzcd}$$

Then T satisfies the commutativity conditions:

- For all X in *FinSet*, and for all nonempty subset $\{x_1, x_2, ..., x_n\} \subset X$,

$$\mu_X \circ \eta_{TX}(\{x_1, x_2, ..., x_n\}) = \mu_X(\{\{x_1, x_2, ..., x_n\}\}) = \{x_1, x_2, ..., x_n\}$$
$$= \bigcup_{1 \leq i \leq n} \{x_i\} = \mu_X\left(\bigcup_{1 \leq i \leq n} \{x_i\}\right) \mu_X \circ T(\eta_X)(\{x_1, x_2, ..., x_n\}).$$

- For all finite sets X, and for all elements V in $TTTX = \mathcal{P}(\mathcal{P}(\mathcal{P}\backslash\{\emptyset\})\backslash\{\emptyset\})\backslash\{\emptyset\}$, on the one hand,

$$\mu_X \circ \mu_{TX}(V) = \mu_X\left(\bigcup_{S \in V} S\right) = \bigcup_{S \in V} \bigcup_{\mathcal{U} \in S} \mathcal{U},$$

  and on the other hand,

$$\mu_X \circ T\mu_X(V) = \mu_X\left(\bigcup_{S \in V} \mu_X(S)\right) = \mu_X\left(\bigcup_{S \in V} \bigcup_{\mathcal{U} \in S} \mathcal{U}\right) = \bigcup_{S \in V} \bigcup_{\mathcal{U} \in S} \mathcal{U}.$$

  We then have

$$\mu \circ (\mu * 1_T) = \mu \circ (1_T * \mu),$$

and T is a monad. $Kl(T)$ is the category of finite sets, with morphisms beeing multivariate morphisms, i.e., $f : X \to Y$ in $Kl(T)$ is a function $f : X \to \mathcal{P}(Y)\backslash\{\emptyset\}$. We call this category *MultiVar*.

Some objects in monoidal categories have interesting properties, they are called *monoids* and *comonoids*.

Definition 2.16. *Monoids and comonoids.* Let C be a symmetric monoidal category. A *monoid* in C is an object X, together with two morphisms:

- a multiplication
$$m : X \otimes X \to X,$$
- and a unit
$$u : I \to X,$$

satisfying associativity,

$$\begin{tikzcd}
X \otimes X \otimes X \arrow[r, "1_X \otimes m"] \arrow[d, "m \otimes 1_X"'] \arrow[dr, phantom, "\circlearrowleft"] & X \otimes X \arrow[d, "m"] \\
X \otimes X \arrow[r, "m"'] & X,
\end{tikzcd}$$

and unitality,

$$\begin{tikzcd}
I \otimes X \arrow[r, "u \otimes 1_X"] \arrow[dr, bend right, "\lambda_X"'] \arrow[r, phantom, shift right=6, "\circlearrowleft"] & X \otimes X \arrow[d, "m"] & X \otimes I. \arrow[l, "1_X \otimes u"'] \arrow[dl, bend left, "\rho_X"] \arrow[l, phantom, shift left=6, "\circlearrowleft"] \\
& X &
\end{tikzcd}$$

A monoid X is *commutative* if in addition, it satisfies

$$\begin{tikzcd}
X \otimes X \arrow[rr, "B_{X,X}"] \arrow[dr, "m"'] & \arrow[d, phantom, "\circlearrowleft"] & X \otimes X. \arrow[dl, "m"] \\
& X &
\end{tikzcd}$$

Dually, a *comonoid* in C is an object X, together with two morphisms:

- a comultiplication
$$cm : X \to X \otimes X,$$
- and a counit
$$cu : X \to I,$$

satisfying coassociativity

$$\begin{tikzcd}
X \arrow[r, "cm"] \arrow[d, "cm"'] \arrow[dr, phantom, "\circlearrowleft"] & X \otimes X \arrow[d, "1_X \otimes cm"] \\
X \otimes X \arrow[r, "cm \otimes 1_X"'] & X \otimes X \otimes X,
\end{tikzcd} \tag{9}$$

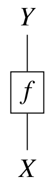
and counitality

$$\begin{tikzcd}
I \otimes X & X \otimes X \arrow[l, "cu \otimes 1_X"'] \arrow[r, "1_X \otimes cu"] & X \otimes I. \\
& X \arrow[ul, bend left, "\lambda_X^{-1}"] \arrow[u, "cm"] \arrow[ur, bend right, "\rho_X^{-1}"'] \arrow[ul, phantom, shift right=6, "\circlearrowleft"] \arrow[ur, phantom, shift left=6, "\circlearrowleft"] &
\end{tikzcd} \tag{10}$$

A comonoid X is *cocommutative* if in addition, it satisfies

$$\begin{tikzcd} X \otimes X \arrow[rr, "B_{X,X}"] & & X \otimes X. \\ & X \arrow[ul, "cm"] \arrow[ur, "cm"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

Notation. We will represent morphisms in *string diagrams*. These diagrams are to be read from the bottom to the top, i.e., for a morphism $f : X \to Y$, we will write



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If it is clear from the context, we will sometimes omit writing the domain and codomain X and Y.

Now we are ready to define the categories that we want to study: The Markov categories.

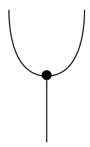
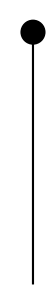
Definition 2.17. [2, 2.1.] A *Markov category* C is a symmetric monoidal category in which every object $X \in C$ is equipped with a commutative comonoid structure given by a comultiplication

$$copy_X : X \to X \otimes X,$$

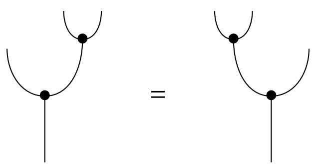
and a counit

$$del_X : X \to I,$$

depicted in string diagrams as

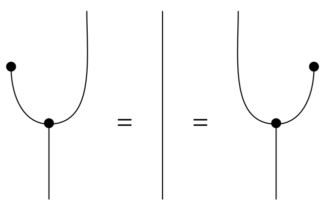
$copy_X =$  , $del_X =$  ,

and satisfying the commutative comonoid equations, i.e., the coassociativity (9) depicted in string diagrams as



, (11)

the counitality (10), depicted in string diagrams as

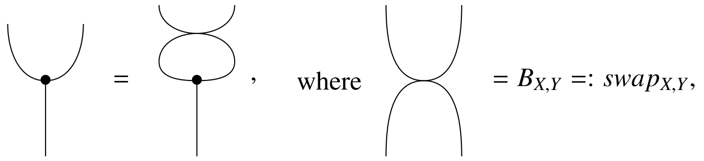


, (12)

and the cocommutativity

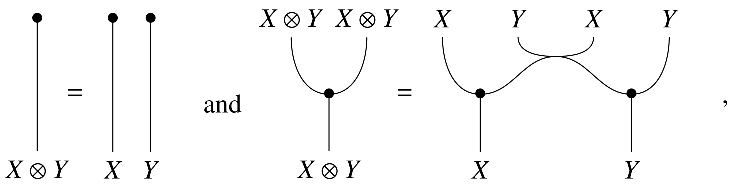
$$\begin{tikzcd} X \arrow[r, "copy_X"] \arrow[d, "copy_X"'] \arrow[dr, phantom, "\circlearrowleft"] & X \otimes X, \\ X \otimes X \arrow[ur, bend right, "B_{X,X}"'] & \end{tikzcd}$$

depicted in string diagrams as



(13)

as well as compatibility with the monoidal structure,



(14)

i.e.,

$$\begin{tikzcd}
X \otimes Y \arrow[r, "del_X \otimes 1_X"] \arrow[d, equal] & I \otimes Y \arrow[r, "1_I \otimes del_Y"] \arrow[d, phantom, "\circlearrowleft"] & I \otimes I \arrow[d, "\lambda_I"] \\
X \otimes Y \arrow[rr, "del_{X\otimes Y}"'] & & I,
\end{tikzcd}$$

and

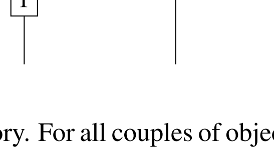
$$\begin{tikzcd}
X \otimes Y \arrow[r, "copy_{X\otimes Y}"] \arrow[d, "copy_X \otimes 1_Y"'] & (X \otimes Y) \otimes (X \otimes Y) \arrow[r, "a^{-1}_{X\otimes Y,X,Y}", "\cong"'] & ((X \otimes Y) \otimes X) \otimes Y \arrow[d, "a_{X,Y,X}\otimes 1_Y", "\cong"'] \\
(X \otimes X) \otimes Y \arrow[d, "1_{X\otimes X}\otimes copy_Y"'] & \circlearrowleft & (X \otimes (Y \otimes X)) \otimes Y \arrow[d, "(1_X \otimes B_{X,Y})\otimes 1_Y"] \\
(X \otimes X) \otimes (Y \otimes Y) \arrow[r, "\cong", "a^{-1}_{X\otimes X,Y,Y}"'] & ((X \otimes X) \otimes Y) \otimes Y \arrow[r, "\cong", "a_{X,X,Y}\otimes 1_Y"'] & (X \otimes (X \otimes Y)) \otimes Y,
\end{tikzcd} \quad (15)$$

we will name the diagonal composite $(X \otimes Y) \otimes (X \otimes Y) \longrightarrow (X \otimes X) \otimes (Y \otimes Y)$ by $gr_{X,Y}$.

C must at last satisfy naturality of deletion, which means that for every morphism f,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "del_X"'] & Y, \arrow[dl, "del_Y"] \\
I &
\end{tikzcd}$$

commutes, which can be depicted in string diagrams as



(16)

Definition 2.18. Let C be a Markov category. For all couples of objects X, Y in C we define the two morphisms,

$$p_X := \rho_X \circ 1_X \otimes del_X : X \otimes Y \longrightarrow X,$$

$$p_Y := \lambda_Y \circ del_Y \otimes 1_Y : X \otimes Y \longrightarrow Y,$$

called projections on X and on Y, respectively. And for all morphisms $f : A \to X \otimes Y$ in C, we define the two morphisms,

$$f_X := p_X \circ f : A \xrightarrow{f} X \otimes Y \xrightarrow{p_X} X,$$

$$f_Y := p_Y \circ f : A \xrightarrow{f} X \otimes Y \xrightarrow{p_Y} Y,$$

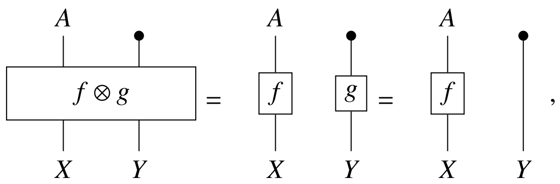
called marginalisations on X and on Y, respectively.

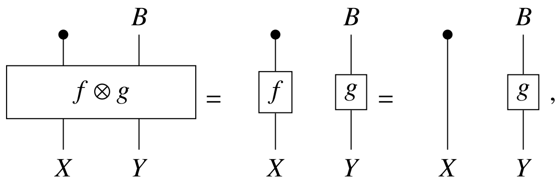
Proposition 2.19. *Let C be a Markov category. There exists two natural transformations $P_1 : (-)_1 \otimes (-)_2 \Rightarrow (-)_1$ and $P_2 : (-)_1 \otimes (-)_2 \to (-)_2$ defined as follows. For all couples of objects X, Y in C,*

$$(P_1)_{X,Y} := p_X : X \otimes Y \longrightarrow X,$$

$$(P_2)_{X,Y} := p_Y : X \otimes Y \longrightarrow Y.$$

Proof. We show that P_1 and P_2 are natural transformations. For all couples of morphisms $f : X \to A$, $g : Y \to B$, by naturality of deletion (16),





and the following diagrams commute,

$$\begin{tikzcd}
X \otimes Y \arrow[r, "f\otimes g"] \arrow[d, "p_X"'] & A \otimes B \arrow[d, "p_A"] & X \otimes Y \arrow[r, "f\otimes g"] \arrow[d, "p_Y"'] & A \otimes B \arrow[d, "p_B"] \\
X \arrow[r, "f"'] & A, & Y \arrow[r, "g"'] & B.
\end{tikzcd}$$

This establishes naturality of P_1 and P_2. $\square$

We can construct new Markov categories from Markov categories we already have. One way to do so arises when we can construct a functor $F : C \to \mathcal{D}$ from a Markov category to an arbitrary category, preserving some properties of the monoidal category. We can then construct the structure morphims for $\mathcal{D}$ to become a Markov category, taking the image in $\mathcal{D}$ of the structure morphisms in C, as follows.

Lemma 2.20. *Let C be a symmetric monoidal category, $\mathcal{D}$ a category with the same class of objects, and $F : C \to \mathcal{D}$ a functor such that*

- *F is the identity on objects;*
- *we can extend $\otimes_C$ to a tensor product $\otimes_{\mathcal{D}}$ on $\mathcal{D}$, i.e., there exists a bifunctor $-\otimes_{\mathcal{D}}- : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ such that for all X, Y in $\mathcal{D}$,*

  $$X \otimes Y := X \otimes_{\mathcal{D}} Y := X \otimes_C Y,$$

  and

  $$F(f) \otimes_{\mathcal{D}} F(g) = F(f \otimes_C g), \tag{17}$$

  for all $f : X \to A$, $g : Y \to B$ in C;
- *F preserves naturality of the associator, the unitors and the braiding of C, i.e.,*

– $F(a) : - \otimes (- \otimes -) \Rightarrow (- \otimes -) \otimes -$ *defined defined pointwise by,*

$$F(a)_{X,Y,Z} := F(a_{X,Y,Z}),$$

for all triples of objects X, Y, Z in $\mathcal{D}$,

– $F(\lambda) : I_{\mathcal{D}} \otimes - \Rightarrow 1_{\mathcal{D}}$ *defined defined pointwise by,*

$$F(\lambda)_X := F(\lambda_X),$$

for all objects X in $\mathcal{D}$,

– $F(\rho) : - \otimes I_{\mathcal{D}} \Rightarrow 1_{\mathcal{D}}$ *defined defined pointwise by,*

$$F(\rho)_X := F(\rho_X),$$

for all objects X in $\mathcal{D}$,

– $F(B)$ *defined defined pointwise by,*

$$F(B)_{X,Y} := F(B_{X,Y}),$$

for all couples of objects X, Y in $\mathcal{D}$,

are natural isomorphisms with respects to all morphisms in $\mathcal{D}$.

Then if we endow $\mathcal{D}$ with the associator, the units and the braiding on $\mathcal{D}$ defined by respectively $F(a)$, $F(\lambda)$, $F(\rho)$ and $F(B)$ defined above, $\mathcal{D}$ becomes a symmetric monoidal category with unit object $I := I_{\mathcal{D}} := I_C$, with respect to which $F : C \to \mathcal{D}$ is a strict symmetric monoidal functor.

If in addition, C is a Markov category and F preserves naturality of deletion, i.e., $F(del)$ defined pointwise by the following,

$$F(del)_X := F(del_X),$$

for all objects X in $\mathcal{D}$ is natural with respects to all morphisms in $\mathcal{D}$. Then we can construct a structure of Markov category on $\mathcal{D}$ with comultiplication $F(copy)$ defined pointwise as follows.

$$F(copy)_X := F(copy_X)$$

for all objects X in $\mathcal{D}$, and deletion $F(del)$ defined above.

Proof.

- *We show $\mathcal{D}$ is a monoidal category.*

1. The bifunctor $\otimes_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$, the unit object, and the structural natural isomorphisms - the associator and the left and right unitors - are defined by assumption.
2. The pentagon identity is satisfied since it is satisfied in C, and for all objects W, X, Y, Z in $\mathcal{D}$,

$$F(a_{W,X,Y}) \otimes_{\mathcal{D}} 1_Z = F(a_{W,X,Y}) \otimes_{\mathcal{D}} F(1_Z) = F(a_{W,X,Y} \otimes_C 1_Z),$$
$$1_W \otimes_{\mathcal{D}} F(a_{X,Y,Z}) = F(1_W) \otimes_{\mathcal{D}} F(a_{X,Y,Z}) = F(1_W \otimes_C a_{X,Y,Z}),$$

by assumption (17), and the identity follows by functoriality of F.
3. The triangle identity holds as well since it is satisfied in C, and for every couple of objects X, Y in $\mathcal{D}$,

$$F(\rho_X) \otimes_{\mathcal{D}} 1_Y = F(\rho_X) \otimes_{\mathcal{D}} F(1_Y) = F(\rho_X \otimes_C 1_Y),$$
$$1_X \otimes_{\mathcal{D}} F(\lambda_Y) = F(1_X) \otimes_{\mathcal{D}} F(\lambda_Y) = F(1_X \otimes_C \lambda_Y),$$

by assumption (17), and the identity follows by functoriality of F.

- *We show that $\mathcal{D}$ is symmetric.* The braiding on $\mathcal{D}$ is defined for all objects X, Y in $\mathcal{D}$ and is natural by assumption. Hexagon identity, unit coherence and symmetry axiom hold by definition of the monoidal structure on $\mathcal{D}$, compatibility of F with the tensor product (17) and functoriality of F.

- *F is a monoidal functor.* Let $\nabla : F(-) \otimes_{\mathcal{D}} F(-) \to F(- \otimes_C -)$ be defined as follows. For all objects X, Y in C,

$$\nabla_{X,Y} := Id_{X \otimes Y} : F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_C Y),$$

which clearly defines a natural transformation, and define the unit by

$$\eta := Id_I : I_{\mathcal{D}} \to F(I_C).$$

With these choices in hand, F respects the monoidal structure since all the morphisms involved in diagrams (1), (2) and (3) are identities or exactly equal by definition of the monoidal structure on $\mathcal{D}$.

- *F is symmetric.* This is direct, since all the morphisms involved in diagram (4) are identities or exactly equal by definition of the monoidal structure on $\mathcal{D}$.
- *We show that F is a strict symmetric monoidal functor.* By definition, ∇ and η are identities.
- Moreover, if C is a Markov category, then *$\mathcal{D}$, together with the structure morphisms of the statement, becomes a Markov category*. By assumption, we can equip every object X in $\mathcal{D}$ with a comultiplication

$$F(copy_X) : X \to X \otimes X,$$

and a counit

$$F(del_X) : X \to I.$$

Then the commutative comonoid equations and compatibility with the monoidal structure are satisfied since F is a strong symmetric monoidal functor with ∇ and η identities, so that the diagrams in $\mathcal{D}$ are exactly the images by F of the diagrams in C, which are satisified since C is a Markov category, and we have naturality of $F(del)$ by assumption.

□

2.2 Standard examples

In this section, we will present three standard examples of Markov categories, *Set*, *FinStoch* and *Gauss*.

Example 2.21. The category of sets, with tensor product defined by the cartesian product is a Markov category.

- *Set is a monoidal category.* The unit object is the set $I := \{0\}$. The associator, the left and right unitor are defined naturally. For every sets X, Y, Z, let

$$\begin{aligned} a_{X,Y,Z} &: (X \times Y) \times Z \to X \times (Y \times Z) \\ &((x, y), z) \mapsto (x, (y, z)) \\ \lambda_X &: I \times X \to X \\ &(0, x) \mapsto x \\ \rho_X &: X \times I \to X \\ &(x, 0) \mapsto x. \end{aligned}$$

They are natural since for all morphisms $f : X \to A$, $g : Y \to B$ and $h : Z \to C$, the diagrams

$$\begin{tikzcd} (X \times Y) \times Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "(f\times g)\times h"'] & X \times (Y \times Z) \arrow[d, "f\times(g\times h)"] \\ (A \times B) \times C \arrow[r, "a_{A,B,C}"] & A \times (B \times C), \end{tikzcd}$$

$$\begin{tikzcd} I \times X \arrow[r, "\lambda_X"] \arrow[d, "1_I\times f"'] & X \arrow[d, "f"] \\ I \times A \arrow[r, "\lambda_A"'] & A, \end{tikzcd}$$

$$\begin{tikzcd} X \times I \arrow[r, "\rho_X"] \arrow[d, "f\times 1_I"'] & X \arrow[d, "f"] \\ A \times I \arrow[r, "\rho_A"'] & A, \end{tikzcd}$$

commute. We can find their inverses, so that they are natural isomorphisms:

$$\begin{aligned} a^{-1}_{X,Y,Z} &: X \times (Y \times Z) \to (X \times Y) \times Z \\ &(x, (y, z)) \mapsto ((x, y), z), \\ \lambda^{-1}_X &: X \to I \times X \\ &x \mapsto (0, x), \\ \rho^{-1}_X &: X \to X \times I \\ &x \mapsto (x, 0). \end{aligned}$$

Now the pentagon and triangle identity follow directly. For every 4-tuple W, X, Y, Z,

$$\begin{aligned}
&a_{W,X,Y\times Z} \circ a_{W\times X,Y,Z}(((w,x),y),z) = a_{W,X,Y\times Z}((w,x),(y,z)) \\
&= (w,(x,(y,z))) \\
&= 1_W \times a_{X,Y,Z}(w,((x,y),z)) = 1_W \times a_{X,Y,Z} \circ a_{W,X\times Y,Z}((w,(x,y)),z) \\
&= 1_W \times a_{X,Y,Z} \circ a_{W,X\times Y,Z} \circ a_{W,X,Y} \times 1_Z(((w,x),y),z),
\end{aligned}$$

and

$$\begin{aligned}
&1_X \times \lambda_Y \circ a_{X,I,Y}((x,0),y) = 1_X \times \lambda_Y(x,(0,y)) \\
&=(x,y) \\
&=\lambda_X \times 1_X((x,0),y).
\end{aligned}$$

Thanks to these identities, it follows that *Set* is a monoidal category.

- *Set is symmetric.* For all sets X, Y, we define the braiding by

$$\begin{aligned}
B_{X,Y} &: X \times Y \to Y \times X \\
&(x,y) \mapsto (y,x).
\end{aligned}$$

It is natural since for all morphisms $f : X \to A$, $g : Y \to B$, the diagram

$$\begin{tikzcd}
X \times Y \arrow[r, "B_{X,Y}"] \arrow[d, "f\times g"'] & Y \times X \arrow[d, "g\times f"] \\
A \times B \arrow[r, "B_{A,B}"'] & B \times A,
\end{tikzcd}$$

commutes. We can define its inverse, so that it is a natural isomorphism:

$$\begin{aligned}
B^{-1}_{X,Y} &: Y \times X \to X \times Y \\
&(y,x) \mapsto (x,y).
\end{aligned}$$

The first hexagon identity is then respected for all set X, Y, Z, and for every $x \in X, y \in Y, z \in Z$:

$$\begin{tikzcd}
((x,y),z) \arrow[r, mapsto, "a_{X,Y,Z}"] \arrow[d, mapsto, "B_{X,Y}\times 1_z"] & (x,(y,z)) \arrow[r, mapsto, "B_{X,Y\times Z}"] & ((y,z),x) \arrow[d, mapsto, "a_{Y,Z,X}"] \\
((y,x),z) \arrow[r, mapsto, "a_{Y,X,Z}"] & (y,(x,z)) \arrow[r, mapsto, "1_Y\times B_{X,Z}"] & (y,(z,x)).
\end{tikzcd}$$

Unit coherence is also respected,

$$\begin{aligned}
\lambda_X \circ B_{X,Y} &: X \times I \to I \times X \to X \\
&(x,0) \mapsto (0,x) \mapsto x \\
= \rho_X &: X \times I \to X \\
&(x,0) \mapsto x.
\end{aligned}$$

The symmetry axiom is also respected,

$$\begin{aligned}
B_{Y,X} \circ B_{X,Y} &: X \times Y \to Y \times X \to X \times Y \\
&(x,y) \mapsto (y,x) \mapsto (x,y) \\
= 1_{X,Y} &: X \times Y \to X \times Y \\
&(x,y) \mapsto (x,y).
\end{aligned}$$

- *Set is a Markov category.* For all sets X, the comultiplication is defined by

$$\begin{aligned}
\Delta_X &: X \to X \times X \\
&x \mapsto (x,x),
\end{aligned}$$

and the counit by the unique morphism $X \to I$

$$\begin{aligned}
del_X &: X \to I \\
&x \mapsto 0.
\end{aligned}$$

Then the commutative comonoid equations are satisfied. First,

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[d, "\Delta_X"'] & X \times X \arrow[d, "1_X \times \Delta_X"] \\
X \times X \arrow[r, "\Delta_X \times 1_X"'] & X \times X \times X,
\end{tikzcd}$$

commutes since for every x in X,

$$\begin{aligned}
&1_X \times \Delta_X \circ \Delta_X(x) = 1_X \times \Delta_X(x, x) \\
&= (x, x, x) \\
&= \Delta_X \times 1_X(x, x) = \Delta_X \times 1_X(x, x) \circ \Delta_X(x).
\end{aligned}$$

Then,

$$\begin{tikzcd}
X \arrow[rr, "\Delta_X"] \arrow[dd, "\Delta_X"'] \arrow[ddrr, equal] & & X \times X \arrow[d, "del_X \times 1_X"] \\
& & I \times X \arrow[d, "\lambda_X"] \\
X \times X \arrow[r, "1_X \times del_X"'] & X \times I \arrow[r, "\rho_X"'] & X,
\end{tikzcd}$$

commutes since for every x in X,

$$\begin{aligned}
&\lambda_X \circ del_X \times 1_X \circ \Delta_X(x) = \lambda_X \circ del_X \times 1_X(x, x) = \lambda_X(0, x) \\
&= x \\
&= \rho_X(x, 0) = \rho_X \circ 1_X \times del_X(x, x) = \rho_X \circ 1_X \times del_X \circ \Delta_X(x).
\end{aligned}$$

And

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[d, "\Delta_X"'] & X \times X, \\
X \times X \arrow[ur, "B_{X,X}"'] &
\end{tikzcd}$$

commutes since for every x in X,

$$B_{X,X} \circ \Delta_X(x) = B_{X,X}(x, x) = (x, x) = \Delta_X(x).$$

We also have compatibility with the monoidal structure,

$$\begin{tikzcd}
X \times Y \arrow[r, "del_{X \times Y}"] \arrow[d, "del_X \times del_Y"'] & I, \\
I \times I \arrow[ur, "\lambda_I"'] &
\end{tikzcd}$$

commutes since for every (x, y) in $X \times Y$,

$$\lambda_I \circ del_X \times del_Y(x, y) = \lambda_I(0, 0) = 0 = del_{X,Y}(x, y).$$

Moreover,

$$\begin{tikzcd}
X \times Y \arrow[d, "\Delta_X \times \Delta_Y"'] \arrow[dr, "\Delta_{X \times Y}"] & \\
X \times X \times Y \times Y \arrow[r, "1_X \times B_{X,Y} \times 1_Y"'] & X \times Y \times X \times Y,
\end{tikzcd}$$

commutes since for every (x, y) in $X \times Y$,

$$1_X \times B_{X,Y} \times 1_Y \circ \Delta_X \times \Delta_Y(x, y) = 1_X \times B_{X,Y} \times 1_Y(x, x, y, y) = (x, y, x, y) = \Delta_{X \times Y}(x, y).$$

Lastly, for all sets X and Y, and every morphism $f : X \to Y$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "del_X"'] & Y \arrow[d, "del_Y"] \\ & I, \end{tikzcd}$$

commutes since I is terminal in *Set*. Thus, we proved that *Set* is a Markov category.

We note that the projections $X \times Y \to X$ and $X \times Y \to Y$ coincide with $p_X = \rho_X \circ (1_X \times del_Y)$ and $p_Y = \lambda_Y \circ (del_Y \times 1_Y)$. This inspired us to define projections on less standard Markov categories in Definition 2.18.

Remark. *FinSet* is also a Markov category, since $I = 0$ is a finite set, and all the structure morphisms defined in *Set* can be defined just as in *FinSet*.

Example 2.22. The category *FinStoch* of finite sets with Markov kernels as morphisms is also a Markov category. *Markov kernels* between finite sets X and Y are morphisms $f : X \to Y$ represented as stochastic matrices $(f_{xy})_{x\in X, y\in Y}$, where the entry $f_{xy} =: f(y|x)$ is interpreted as the probability that the Markov kernel f returns the value y on input x. So we impose the stochastic condition, i.e., for every x in X,

$$\sum_{y\in Y} f(y|x) = 1.$$

Composition of morphisms is given by matrix multiplication, so that $g \circ f$ with $g : Y \to Z$ turns into the Chapman-Kolmogorov equation,

$$(g \circ f)(z|x) = \sum_{y\in Y} g(z|y) f(y|x), \quad \forall x \in X, z \in Z.$$

The composition is associative, since for all triples of morphisms $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, for all w in W and z in Z,

$$\begin{aligned} ((h \circ g) \circ f)(z|w) &= \sum_{x\in X} (h \circ g)(z|x) f(x|w) \\ &= \sum_{x\in X} \sum_{y\in Y} h(z|y) g(y|x) f(x|w) \\ &= \sum_{y\in Y} h(z|y)(g \circ f)(y|w) \\ &= (h \circ (g \circ f))(z|w). \end{aligned}$$

The identities are the morphisms

$$1_X(x'|x) := \begin{cases} 1 & \text{if } x' = x \\ 0 & \text{otherwise.} \end{cases} = \delta_{x',x} : X \to X,$$

and we have, for every morphism $f : X \to Y$ in *FinStoch*,

$$\begin{aligned} f \circ 1_X(y|x) &= \sum_{x'\in X} f(y|x') \cdot \delta_{x',x} = f(y|x), \\ 1_Y \circ f(y|x) &= \sum_{y'\in Y} \delta_{y,y'} \cdot f(y'|x) = f(y|x), \end{aligned}$$

so that *FinStoch* is a category. The tensor product is given on objects by cartesian product on sets. On morphisms, let $f : A \to X$ and $g : B \to Y$, then

$$(f \otimes g)((x, y)|(a, b)) := f(x|a) g(y|b).$$

There exists a functor $F : FinSet \to FinStoch$, acting as the identity on objects. On morphisms, for $f : X \to Y$ in *FinSet*, we can define $\tilde{f} := F(f) : X \to Y$, considering that for an input x in X, we are sure to assign one output $y = f(x)$ in Y :

$$\tilde{f}(y|x) = \begin{cases} 1 & \text{if } y = f(x) \\ 0 & \text{otherwise.} \end{cases} = \delta_{y,f(x)}.$$

The composition is preserved since if $f : X \to Y$ and $g : Y \to Z$ are morphisms in *FinSet*, then if we take their representation in *FinStoch* and then compose, we get the same morphism as if we directly took the representation of their composition $g \circ f$:

$$\begin{aligned} \tilde{g}(z|y)\tilde{f}(y|x) &= \begin{cases} 1 & \text{if } z = g(y) \text{ and } y = f(x) \\ 0 & \text{otherwise} \end{cases} \\ \implies \tilde{g} \circ \tilde{f}(z|x) = \sum_{y \in Y} \tilde{g}(z|y)\tilde{f}(y|x) &= \begin{cases} 1 & \text{if } z = g(f(x)) \\ 0 & \text{otherwise} \end{cases} = \widetilde{g \circ f}. \end{aligned} \tag{18}$$

For all sets X, and for all x, x' in X,

$$\widetilde{Id_X}(x'|x) = \delta_{x,x'} = 1_X(x'|x).$$

Remark. For all morphisms $f : X \to Y$ in *FinStoch*, $g : Y \to Z$ and $h : W \to X$ in *FinSet*,

$$\tilde{g} \circ f(z|x) = \sum_{y \in Y} \delta_{g(y),z} \cdot f(y|x) = \sum_{y \in g^{-1}(z)} f(y|x), \tag{19}$$

and

$$f \circ \tilde{h}(y|w) = \sum_{x \in X} f(y|x) \cdot \delta_{h(w),x} = f(y|h(w)). \tag{20}$$

The conditions of Lemma 2.20 are satisfied since

- The functor F is the identity on objects.
- For all objects X, Y in *FinStoch*,

  $$X \otimes Y = X \times Y,$$

  and for all morphisms $f : X \to A$, $g : Y \to B$ in *Set*, we can interpret their product $\tilde{f} \otimes \tilde{g} : X \otimes Y \to A \otimes B$ in *FinStoch* in two equal ways:

  $$\begin{aligned} F(f \times g)((a,b)|(x,y)) = (\widetilde{f \times g})((a,b)|(x,y)) &= \begin{cases} 1 & \text{if } (a,b) = (f(x), g(y)) \\ 0 & \text{otherwise} \end{cases} \\ &= \delta_{(a,b),(f(x),f(y))} \\ &= \delta_{a,f(x)} \cdot \delta_{b,f(y)} \\ &= (\tilde{f} \otimes \tilde{g})((a,b)|(x,y)) = (F(f) \otimes F(g))((a,b)|(x,y)). \end{aligned} \tag{21}$$

- *The functor F preserves naturality of the associator, the unitors, the braiding and deletion of C.*
  - *The image of the associator a on* FinSet *is natural.* For all morphisms $f : X \to A$, $g : Y \to B$ and $h : Z \to C$ in *FinStoch*,

    $$F(a_{X,Y,Z})((x',(y',z'))|((x,y),z)) = \begin{cases} 1 & \text{if } ((x',y'),z') = ((x,y),z) \\ 0 & \text{otherwise,} \end{cases}$$

    $$F(a_{A,B,C})((a',(b',c'))|((a,b),c)) = \begin{cases} 1 & \text{if } ((a',b'),c') = ((a,b),c) \\ 0 & \text{otherwise,} \end{cases}$$

    so that, by (19) and (20),

    $$\begin{aligned} &F(a_{A,B,C}) \circ (f \otimes g) \otimes h((a,(b,c))|((x,y),z)) \\ =&(f \otimes g) \otimes h(((a,b),c)|((x,y),z)) \\ =&f(a|x) \cdot g(b|y) \cdot h(c|z) \\ =&f \otimes (g \otimes h)((a,(b,c))|(x,(y,z))) \\ =&f \otimes (g \otimes h) \circ F(a_{X,Y,Z})((a,(b,c))|((x,y),z)), \end{aligned}$$

    and the following diagram commutes

    $$\begin{tikzcd} (X \otimes Y) \otimes Z \arrow[r, "F(a_{X,Y,Z})"] \arrow[d, "(f\otimes g)\otimes h"'] & X \otimes (Y \otimes Z) \arrow[d, "f\otimes(g\otimes h)"] \\ (A \otimes B) \otimes C \arrow[r, "F(a_{A,B,C})"'] & A \otimes (B \otimes C), \end{tikzcd}$$

    establishing naturality of $F(a)$.

– *The images in FinStoch of the left and right unitors λ and ρ in FinSet are natural.* For every morphism $f : X \to Y$ in *FinStoch*, since

$$\begin{aligned}F(\lambda_X)(x'|(0,x)) &= \delta_{x',x},\\ F(\lambda_Y)(y'|(0,y)) &= \delta_{y',y},\\ F(\rho_X)(x'|(x,0)) &= \delta_{x',x},\\ F(\rho_Y)(y'|(y,0)) &= \delta_{y',y},\end{aligned}$$

we have by (19) and (20),

$$\begin{aligned}&F(\lambda_Y)\circ(1_I\otimes f)(y|(0,x))\\ =&(1_I\otimes f)((0,y)|(0,x))\\ =&1_I(0,0)\cdot f(y|x)\\ =&f(y|x)\\ =&f\circ F(\lambda_X)(y|(0,x)),\end{aligned}$$

dually,

$$F(\rho_Y)\circ(f\otimes 1_Y)(y|(x,0)) = f(y|x) = f\circ F(\rho_X)(y|(x,0)),$$

and the following diagram commutes

$$\begin{tikzcd} I\otimes X \arrow[r, "F(\lambda_X)"] \arrow[d, "1_I\otimes f"'] & X \arrow[d, "f"] & X\otimes I \arrow[l, "F(\rho_X)"'] \arrow[d, "f\otimes 1_I"] \\ I\otimes Y \arrow[r, "F(\lambda_Y)"'] & Y & Y\otimes I, \arrow[l, "F(\rho_Y)"] \end{tikzcd}$$

establishing naturality of $F(\lambda)$ and $F(\rho)$.

– *The image of the braiding B in* FinSet *is natural.* For all morphisms $f : X \to A$ and $g : Y \to B$ in *FinStoch*, since

$$F(B_{X,Y})((y',x')|(x,y)) = \begin{cases} 1 & \text{if } (y',x') = (y,x)\\ 0 & \text{otherwise,}\end{cases}$$

$$F(B_{A,B})((a',b')|(a,b)) = \begin{cases} 1 & \text{if } (b',a') = (b,a)\\ 0 & \text{otherwise,}\end{cases}$$

we have by (19) and (20), for all a in A, b in B, x in X and y in Y,

$$\begin{aligned}&F(B_{A,B})\circ(f\otimes g)((b,a)|(x,y))\\ =&(f\otimes g)((a,b)|(x,y))\\ =&f(a|x)\cdot g(b|y) = g(b|y)\cdot f(a|x)\\ =&(g\otimes f)((b,a)|(y,x))\\ =&(g\otimes f)\circ F(B_{X,Y})((b,a)|(x,y)),\end{aligned}$$

and the following diagram commutes

$$\begin{tikzcd} X\otimes Y \arrow[r, "F(B_{X,Y})"] \arrow[d, "f\otimes g"'] & Y\otimes X \arrow[d, "g\otimes f"] \\ A\otimes B \arrow[r, "F(B_{A,B})"'] & B\otimes A, \end{tikzcd}$$

establishing naturality of $F(B)$.

– *The image of deletion by F is natural.* For every morphism $f : X \to Y$ in *FinStoch*, since,

$$\begin{aligned}F(del_X)(0|x) &= 1,\\ F(del_Y)(0|y) &= 1,\end{aligned}$$

for all x in X and y in Y, so that we have, for all x in X,

$$\begin{aligned}&(F(del_Y)\circ f)(0|x)\\ =&\sum_{y\in Y} f(y|x)\\ =&1\\ =&del_X(0|x)\end{aligned}$$

and the following diagram commutes,

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "F(del_X)"'] & & Y \arrow[dl, "F(del_Y)"] \\
& I, &
\end{tikzcd}$$

establishing naturality of $F(del)$.

Since the conditions of Lemma 2.20 are satisfied, we can conclude that *FinStoch* is a Markov category, and F : *FinSet* $\to$ *FinStoch* is a strong symmetric monoidal functor. The comonoid operations are defined on *FinStoch* by

$$\begin{aligned}
&\widetilde{del}_X : X \to I \\
&\widetilde{del}(0|x) = 1, && \forall x \in X, \\
&\widetilde{copy}_X : X \to X \times X \\
&\widetilde{copy}_X(x_1, x_2|x_0) = \begin{cases} 1 & \text{if } x_0 = x_1 = x_2 \\ 0 & \text{otherwise,} \end{cases} && \forall x_0, x_1, x_2 \in X.
\end{aligned}$$

for all objects X in *FinStoch*.

Remark. As in *Set*, there are projection-type morphisms $\tilde{p}_X : X \otimes Y \to X$ and $\tilde{p}_Y : X \otimes Y \to Y$ defined by

$$\begin{aligned}
\tilde{p}_X(x'|(x,y)) &= \rho_X \circ \tilde{1}_X \otimes \widetilde{del}_Y(x'|(x,y)) \\
&= \delta_{x,x'},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{p}_Y(y'|(x,y)) &= \lambda_Y \circ \widetilde{del}_X \otimes \tilde{1}_Y(y'|(x,y)) \\
&= \delta_{y,y'}.
\end{aligned}$$

We remark that if we have a morphism $f : A \to X \times Y$, interpreted as the probality for x, y in $X \times Y$ knowing a in A, we can marginalise the probabilities on X and on Y using the composition with $\tilde{p}_X$ and $\tilde{p}_y$:

$$\begin{aligned}
f_{|X}(x|a) &:= (\tilde{p}_X \circ f)(x|a) \\
&= \sum_{(x',y)\in X\times Y} \delta_{x,x'} \cdot f((x',y)|a) \\
&= \sum_{y\in Y} f((x,y)|a),
\end{aligned}$$

$$\begin{aligned}
f_{|Y}(y|a) &:= (\tilde{p}_Y \circ f)(y|a) \\
&= \sum_{(x,y')\in X\times Y} \delta_{y,y'} \cdot f((x,y')|a) \\
&= \sum_{x\in X} f((x,y)|a).
\end{aligned}$$

This coincides with the usual marginalisation in probability theory.

Example 2.23. In [2], Fritz defined the category *Gauss* with objects the elements of $\mathbb{N}$ and morphisms f : $n \to m$ defined by a triple (M, C, s), with $M \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{m\times m}$ positive semidefinite, and $s \in \mathbb{R}^m$. The idea is to think of this morphism as a conditional distribution of a random variable $Y \in \mathbb{R}^m$ in terms of $X \in \mathbb{R}^n$ as

$$Y := MX + \xi,$$

with ξ a Gaussian noise independent of X, with mean $\mathbf{E}[\xi] := s$ and covariance matrix $\mathbf{Var}[\xi] = C$. If C is invertible, then the distribution of Y has density with respect to Lebesgue measure on $\mathbb{R}^m$ given by

$$\frac{1}{\sqrt{(2\pi)^m det(C)}} exp\left(-\frac{1}{2}(y - Mx - s)^T C^{-1}(y - Mx - s)\right),$$

i.e. $(Y|X) \sim \mathcal{N}(MX + s, C)$.

We claim that we can equip *Gauss* with a structure of a Markov category.

Proof. • *Gauss is a category.* We want to define the composition of two morphisms $(M, C, s) : n \to m$ and $(N, D, t) : m \to l$. If we go back to the probability interpretation, we can represent these two morphisms by the random variables $Y = MX + \xi$ and $Z = NY + \eta$, where $\xi \sim \mathcal{N}(t, D)$ and $\eta \sim \mathcal{N}(s, C)$. If we compose them, then we should get the random variable Z in terms of X, and we would like to reassociate that to a morphism in *Gauss*

$$Z = NMX + \underbrace{N\xi + \eta}_{\text{Gaussian noise}}.$$

We need to find the distribution of the new Gaussian noise. Assuming that the noises ξ and η are independent of each other, we can compute the expectation

$$\mathbf{E}[N\xi + \eta] = N\mathbf{E}[\xi] + \mathbf{E}[\eta] = Ns + t,$$

and the covariance matrix

$$\mathbf{Var}[N\xi + \eta] = N\mathbf{Var}[\xi] + \mathbf{Var}[\eta] = NCN^T + D.$$

This gives us the definition for the composition :

$$(N, D, t) \circ (M, C, s) := (NM, NCN^T + D, Ns + t),$$

which is associative. If $(N, D, t) : n \to m$, $(M, C, s) : m \to l$ and $(L, B, r) : l \to k$, then

$$\begin{aligned}
(L, B, r) \circ ((M, C, s) \circ (N, D, t)) &= (L, B, r) \circ (MN, MDM^T + C, Mt + s) \\
&= (LMN, L(MDM^T + C)L^T + B, L(Mt + s) + r) \\
&= (LMN, LMD(LM)^T + LCL^T + B, LMt + Ls + r) \\
&= (LM, LCL^T + B, Ls + r) \circ (N, D, t) \\
&= ((L, B, r) \circ (M, C, s)) \circ (N, D, t).
\end{aligned}$$

The identity morphisms are $(1_n, 0, 0) : n \to n$, with 1_n the identity matrix in $\mathbb{R}^n$: for every $f = (N, D, t) : n \to m$

$$\begin{aligned}
(1_m, 0, 0) \circ (N, D, t) &= (1_m N, 1_m D 1_m + 0, 1_m t + 0) = (N, D, t), \\
(N, D, t) \circ (1_n, 0, 0) &= (N 1_n, N0N^T + D, N0 + t) = (N, D, t),
\end{aligned}$$

i.e., the diagram

$$\begin{tikzcd}
n \arrow[r, "f"] \arrow[d, "I_n"'] \arrow[dr, "f"] & m \arrow[d, "I_m"] \\
n \arrow[r, "f"'] & m,
\end{tikzcd}$$

commutes. Note that this choice of identity is consistent with the probability interpretation, since those identity morphisms give random variables $Y \in \mathbb{R}^n$ in terms of $X \in \mathbb{R}^n$:

$$Y = X + \xi,$$

with $\xi \sim \mathcal{N}(0, 0)$ so that $(Y|X) \sim \mathcal{N}(X, 0)$, this means that we are sure to find $Y = X$ since the covariance matrix is 0.

• *Gauss is a monoidal category.*

1. It is a category from the previous point.
2. The tensor product is defined on objects n, m in $\mathbb{N}$ by :

$$n \otimes m := n + m.$$

On morphisms, if $f : n \to m$ and $g : k \to l$ are defined by (M, C, s) and (N, D, t), we want to define define their product $f \otimes g : n + k \to m + l$. By $- \oplus -$ we will mean the direct sum of two matrices, i.e., for $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^l$,

$$X \oplus Y := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \mathbb{R}^{m+l}$$

Let us think of it as a random variable $X \oplus Y \in \mathbb{R}^{m+l}$ with X in $\mathbb{R}^m$ and Y in $\mathbb{R}^l$ in terms of $A \oplus B \in \mathbb{R}^{n+k}$ with A in $\mathbb{R}^n$ and B in $\mathbb{R}^k$, such that

$$X = MA + \xi$$
$$Y = NB + \eta,$$

where $\xi \sim \mathcal{M}(s, C)$ and $\eta \sim \mathcal{N}(t, D)$. So we would like to have

$$X \oplus Y = (M \oplus N)(A \oplus B) + \xi \oplus \eta,$$

where $\xi \oplus \eta \sim \mathcal{N}(s \oplus t, C \oplus D)$. So we can define the tensor product by

$$(M, C, s) \otimes (N, D, t) := (M \oplus N, C \oplus D, s \oplus t).$$

We show that $- \otimes -$ is a bifunctor as follows. If $f_1 : n \to m'$, $f_2 : m' \to m$, $g_1 : k \to l'$ and $g_2 : l' \to l$ are defined respectively by (M_1, C_1, s_1) (M_2, C_2, s_2), (N_1, D_1, t_1), and (N_2, D_2, t_2), then we want to show that

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \otimes g_1).$$

This holds since,

$$\begin{aligned}
&((M_2, C_2, s_2) \otimes (N_2, D_2, t_2)) \circ ((M_1, C_1, s_1) \otimes (N_1, D_1, t_1)) \\
=&(M_2 \oplus N_2, C_2 \oplus D_2, s_2 \oplus t_2) \circ (M_1 \oplus N_1, C_1 \oplus D_1, s_1 \oplus t_1) \\
=&((M_2 \oplus N_2)(M_1 \oplus N_1), (M_2 \oplus N_2)(C_1 \oplus D_1)(M_2 \oplus N_2)^T + C_2 \oplus D_2, \\
&(M_2 \oplus N_2)(s_1 \oplus t_1) + (s_2 \oplus t_2)) \\
=&((M_2M_1) \oplus (N_2N_1), (M_2C_1M_2^T + C_2) \oplus (N_2D_1N_2^T + D_2), (M_2s_1 + s_2) \oplus (N_2t_1 + t_2)) \\
=&(M_2M_1, M_2C_1M_2^T + C_2, M_2s_1 + s_2) \otimes (N_2N_1, N_2D_1N_2^T + D_2, N_2t_1 + t_2) \\
=&((M_2, C_2, s_2) \circ (M_1, C_1, s_1)) \otimes ((N_2, D_2, t_2) \circ (N_1, D_1, t_1)).
\end{aligned}$$

3. The unit object is defined by $I := 0$.
4. For every triple l, m, n of objects, we define the associator $a_{l,m,n} : (l \otimes m) \otimes n \to l \otimes (m \otimes n)$ by the identity $(1_{l+m+n}, 0, 0)$ since $(l \otimes m) \otimes n = l \otimes (m \otimes n) = l + m + n$. This defines a natural isomorphism.
5. For every object n, we define the left unitor $\lambda_n : I \otimes n \to n$ by the identity $(1_n, 0, 0)$ since $I \otimes n = 0 + n = n$. This defines a natural isomorphism.
6. Dually, for every object n, we define the left unitor $\rho_n : n \otimes I \to n$ by the identity $(1_n, 0, 0)$ since $n \otimes I = n + 0 = n$. This defines a natural isomorphism.

The pentagon and triangle identities hold directly, since all the morphisms involved are identities. This yields that *Gauss* is a monoidal category.

- *Gauss is symmetric.* For every $n, m \in \mathbb{N}$, define the braiding $B_{n,m} : n \otimes m \to m \otimes n$ by

$$\left(\begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix}, 0, 0\right),$$

$$B_{n,m}^{-1} = \left(\begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix}^{-1}, 0, 0\right) = \left(\begin{pmatrix} 0 & 1_n \\ 1_m & 0 \end{pmatrix}, 0, 0\right) = B_{m,n}.$$

The hexagon identity holds since for every $l, m, n \in \mathbb{N}$,

$$1_m \otimes B_{l,n} \circ B_{l,m} \otimes 1_n = \left(\begin{pmatrix} 1_m & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & 1_l & 0 \end{pmatrix}, 0, 0\right) \circ \left(\begin{pmatrix} 0 & 1_m & 0 \\ 1_l & 0 & 0 \\ 0 & 0 & 1_n \end{pmatrix}, 0, 0\right) = \left(\begin{pmatrix} 0 & 1_{m+n} \\ 1_l & 0 \end{pmatrix}, 0, 0\right) = B_{l,m\otimes n}.$$

The unit coherence and the symmetry follow directly since all the morphisms involved are identities. This yields that *Gauss* is a symmetric monoidal category.

- *Gauss is a Markov category.* For every $n \in \mathbb{N}$, we can define a comultiplication $copy_n : n \to 2n$,

$$copy_n := \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right),$$

and a counit $del_n : n \to 0$,

$$del_n := (0_{0\times n}, 0_{0\times 0}, 0_{0\times 0}),$$

where $0_{0\times n}$ and $0_{0\times 0}$ are matrices of dimension $0 \times n$ and 0×0. By this, we mean that for every M in $\mathbb{R}^{m\times n}$,

$$\begin{aligned}
&0_{0\times m}M = 0_{0\times n},\\
&M \oplus 0_{0\times l} = \begin{pmatrix} M & 0 \end{pmatrix} \in \mathbb{R}^{m\times(n+l)},\\
&0_{0\times l} \oplus M = \begin{pmatrix} 0 & M \end{pmatrix} \in \mathbb{R}^{m\times(l+n)},\\
&0_{0\times n} \oplus 0_{0\times m} = 0_{0\times(n+m)},\\
&0_{0\times 0}0_{0\times m} = 0_{0\times 0},\\
&M \oplus 0_{0\times 0} = M = 0_{0\times 0} \oplus M,
\end{aligned}$$

and $0_{0\times n}^T = 0_{n\times 0}$. We can define similar conditions on $0_{n\times 0}$ dually.

The morphisms defined above satisfy the comonoid equations. For every $n \in \mathbb{N}$, coassociativity holds, since

$$\begin{aligned}
&(1_n \otimes copy_n) \circ copy_n\\
&\left(1_n \otimes \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\right) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&\left(\begin{pmatrix} 1_n & 0 \\ 0 & 1_n \\ 0 & 1_n \end{pmatrix}, 0, 0\right) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&\left(\begin{pmatrix} 1_n \\ 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&\left(\begin{pmatrix} 1_n & 0 \\ 1_n & 0 \\ 0 & 1_n \end{pmatrix}, 0, 0\right) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&\left(\left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right) \otimes 1_n\right) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&(copy_n \otimes 1_n) \circ copy_n.
\end{aligned}$$

Counitality holds as well, since

$$\begin{aligned}
&\lambda_n \circ (del_n \otimes 1_n) \circ copy_n\\
=&(1_n, 0, 0) \circ ((0_{0\times n}, 0_{0\times 0}, 0_{0\times 0}) \otimes (1_n, 0, 0)) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&(1_n, 0, 0) \circ \left(\begin{pmatrix} 0 & 1_n \end{pmatrix}, 0, 0\right) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&1_n\\
=&(1_n, 0, 0) \circ \left(\begin{pmatrix} 1_n & 0 \end{pmatrix}, 0, 0\right) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&(1_n, 0, 0) \circ ((1_n, 0, 0) \otimes (0_{0\times n}, 0_{0\times 0}, 0_{0\times 0})) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)\\
=&\rho_X \circ (1_n \otimes del_n) \circ copy_n.
\end{aligned}$$

And cocommutativity is also respected, since

$$B_{n,n} \circ copy_n = \left(\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, 0, 0\right) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right) = \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right) = copy_n.$$

Moreover, we have compatibility with the monoidal structure,

$$del_{n\otimes m} = (0_{0\times(n+m)}, 0_{0\times 0}, 0_{0\times 0}) = del_n \otimes del_m,$$

$$
\begin{aligned}
&(1_n \otimes B_{n,m} \otimes 1_m) \circ (copy_n \otimes copy_m)\\
=&\left(\left(1_n \oplus \begin{pmatrix} 0 & 1_m \\ 1_n & 0 \end{pmatrix} \oplus 1_m\right) \cdot \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix} \oplus \begin{pmatrix} 1_m \\ 1_m \end{pmatrix}\right), 0, 0\right)\\
=&\left(\begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 0 & 1_m & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & 0 & 1_m \end{pmatrix} \cdot \begin{pmatrix} 1_n & 0 \\ 1_n & 0 \\ 0 & 1_m \\ 0 & 1_m \end{pmatrix}, 0, 0\right)\\
=&\left(\begin{pmatrix} 1_n & 0 \\ 0 & 1_m \\ 1_n & 0 \\ 0 & 1_m \end{pmatrix}, 0, 0\right)\\
=&\left(\begin{pmatrix} 1_{n+m} \\ 1_{n+m} \end{pmatrix}, 0, 0\right)\\
=&copy_{n\otimes m}.
\end{aligned}
$$

Finally, the naturality of deletion holds also in *Gauss*. For every $f : n \to m$ defined by (M, C, s),

$$del_m \circ f = (0_{0\times m}, 0_{0\times 0}, 0_{0\times 0}) \circ (M, C, s) = (0_{0\times n}, 0_{0\times 0}, 0_{0\times 0}) = del_n.$$

This yields that *Gauss* is a Markov category.

□

2.3 Markov categories from Kleisli categories

As we mentionned above, one can construct a new Markov category from a monad on an old Markov category. In this section, we will see how it works.

Proposition 2.24. *[2, 3.1]. Let $\mathcal{D}$ be a symmetric monoidal category and $T : \mathcal{D} \to \mathcal{D}$ a symmetric monoidal monad with multiplication $\mu : TT \to T$ and unit $\eta : I \to T$ equipped with the morphism $\nabla_{X,Y} : TX \otimes TY \to T(X \otimes Y)$. Then, the Kleisli category $Kl(T)$ becomes symmetric monoidal, with:*

- *the tensor product of objects being the one of $\mathcal{D}$,*
- *the tensor product of morphisms represented by $f : A \to TX$ and $g : B \to TY$ being represented by the composite*

$$A \otimes B \xrightarrow{f\otimes g} TX \otimes TY \xrightarrow{\nabla_{X,Y}} T(X \otimes Y).$$

Moreover, the standard inclusion functor $\mathcal{D} \to Kl(T)$ is strict symmetric monoidal.

Proof. We want to apply Lemma 2.20 to the standard inclusion functor $F : \mathcal{D} \to Kl(T)$. We now show that $- \otimes_{Kl(T)} -$ is indeed a bifunctor. Recall that the identity on A in $Kl(T)$ is η_A, and that the diagram

$$
\begin{tikzcd}
A \otimes_{Kl} B \arrow[r, "\eta_A \otimes_{\mathcal{D}} \eta_B"] \arrow[d, "\eta_{A\otimes_{Kl} B}"'] & T(A) \otimes_{Kl} T(B), \arrow[dl, "\nabla_{A,B}"] \\
T(A \otimes_{Kl} B) &
\end{tikzcd}
$$

commutes by diagram (8) in Definition 2.8. We still have to prove that composition is preserved.

Let $f : A \to X$, $g : B \to Y$, $f' : X \to C$ and $g' : Y \to E$ morphisms in $Kl(T)$ together with their representatives in $\mathcal{D}$ $\underline{f} : A \to TX$, $\underline{g} : B \to TY$, $\underline{f'} : X \to TC$, $\underline{g'} : Y \to TE$, we want to prove that in $\mathcal{D}$,

$$\underline{(f' \otimes_{Kl(T)} g') \circ (f \otimes_{Kl(T)} g)} = \underline{(f' \circ f) \otimes_{Kl(T)} (g' \circ g)}.$$

We have that

$$
\begin{tikzcd}
A \otimes_{\mathcal{D}} B \arrow[r, "f\otimes_{Kl} g"] \arrow[d, "\underline{f}\otimes_{\mathcal{D}}\underline{g}"'] & T(X \otimes_{\mathcal{D}} Y), \\
TX \otimes_{\mathcal{D}} TY \arrow[ur, "\nabla_{X,Y}"'] &
\end{tikzcd}
$$

commutes by definition of the tensor product in $Kl(T)$ on morphisms,

$$\begin{tikzcd}
 & T(X \otimes_{\mathcal{D}} Y) \arrow[d, "T(\underline{f'} \otimes_{\mathcal{D}} \underline{g'})"] \\
 & T(TC \otimes_{\mathcal{D}} TE) \\
TX \otimes_{\mathcal{D}} TY \arrow[uur, "\nabla_{X,Y}"] \arrow[r, "T(\underline{f'}) \otimes_{\mathcal{D}} T(\underline{g'})"'] & TTC \otimes_{\mathcal{D}} TTE, \arrow[u, "\nabla_{TC,TE}"']
\end{tikzcd}$$

commutes by naturality of ∇,

$$\begin{tikzcd}
T(X \otimes_{\mathcal{D}} Y) \arrow[r, "T(\underline{f' \otimes_{Kl} g'})"] \arrow[d, "T(\underline{f'} \otimes_{\mathcal{D}} \underline{g'})"'] & TT(C \otimes_{\mathcal{D}} E), \\
T(TC \otimes_{\mathcal{D}} TE) \arrow[ur, "T\nabla_{C,E}"'] &
\end{tikzcd}$$

commutes by definition of the tensor product on morphisms in $Kl(T)$,

$$\begin{tikzcd}
 & TT(C \otimes_{\mathcal{D}} E) \arrow[d, "\mu_{C \otimes_{\mathcal{D}} E}"] \\
T(TC \otimes_{\mathcal{D}} TE) \arrow[ur, "T\nabla_{C,E}"] & T(C \otimes_{\mathcal{D}} E) \\
TTC \otimes_{\mathcal{D}} TTE \arrow[u, "\nabla_{TC,TE}"] \arrow[r, "\mu_C \otimes_{\mathcal{D}} \mu_E"'] & TC \otimes_{\mathcal{D}} TE, \arrow[u, "\nabla_{C,E}"']
\end{tikzcd}$$

commutes by the diagram (7), from the definition of a symmetric monoidal monad. Putting together all these commuting diagrams, we get the big commuting diagram

$$\begin{tikzcd}
A \otimes_{\mathcal{D}} B \arrow[r, "\underline{f \otimes_{Kl} g}"] \arrow[ddd, "\underline{f} \otimes_{\mathcal{D}} \underline{g}"'] \arrow[dr, phantom, "\circlearrowleft"] & T(X \otimes_{\mathcal{D}} Y) \arrow[r, "T(\underline{f' \otimes_{Kl} g'})"] \arrow[dd, "T(\underline{f'} \otimes_{\mathcal{D}} \underline{g'})"] \arrow[dr, phantom, "\circlearrowleft"] & TT(C \otimes_{\mathcal{D}} E) \arrow[d, "\mu_{C \otimes_{\mathcal{D}} E}"] \\
 & & T(C \otimes_{\mathcal{D}} E) \\
 & T(TC \otimes_{\mathcal{D}} TE) \arrow[uur, bend right, "T\nabla_{C,D}"'] \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
TX \otimes_{\mathcal{D}} TY \arrow[uuur, "\nabla_{X,Y}"] \arrow[r, "T(\underline{f'}) \otimes_{\mathcal{D}} T(\underline{g'})"'] & TTC \otimes_{\mathcal{D}} TTE \arrow[u, "\nabla_{TC,TE}"] \arrow[r, "\mu_C \otimes_{\mathcal{D}} \mu_E"'] & TC \otimes_{\mathcal{D}} TE. \arrow[uu, "\nabla_{C,E}"']
\end{tikzcd} \tag{22}$$

By definition of composition in $Kl(T)$, going first down, then right in the diagram above, we have:

$$(\mu_C \circ T(\underline{f'}) \circ \underline{f}) \otimes_{\mathcal{D}} (\mu_E \circ T(\underline{g'}) \circ \underline{g}) = (\underline{f' \circ f}) \otimes_{\mathcal{D}} (\underline{g' \circ g}).$$

By bifunctoriality of $- \otimes_{\mathcal{D}} -$ and definition of composition and product in $Kl(T)$, the following diagrams commute:

$$\begin{tikzcd}
A \otimes_{\mathcal{D}} B \arrow[r, "\underline{(f' \otimes_{Kl(T)} g') \circ (f \otimes_{Kl(T)} g)}"] \arrow[d, "\underline{f \otimes_{Kl} g}"'] & T(C \otimes_{\mathcal{D}} E) \\
T(X \otimes_{\mathcal{D}} Y) \arrow[r, "T(\underline{f' \otimes_{Kl} g'})"'] & TT(C \otimes_{\mathcal{D}} E) \arrow[u, "\mu_{C \otimes_{\mathcal{D}} E}"']
\end{tikzcd}
,
\begin{tikzcd}
A \otimes_{\mathcal{D}} B \arrow[r, "(\underline{f' \circ f}) \otimes_{\mathcal{D}} (\underline{g' \circ g})"] \arrow[dr, "\underline{(f' \circ f) \otimes_{Kl(T)} (g' \circ g)}"'] \arrow[dd, bend right, "\underline{f} \otimes_{\mathcal{D}} \underline{g}"'] & T(C \otimes_{\mathcal{D}} E) \\
 & TC \otimes_{\mathcal{D}} TE \arrow[u, "\nabla_{C,E}"'] \\
TX \otimes_{\mathcal{D}} TY \arrow[r, "T(\underline{f'}) \otimes_{\mathcal{D}} T(\underline{g'})"'] & TTC \otimes_{\mathcal{D}} TTE \arrow[u, "\mu_C \otimes_{\mathcal{D}} \mu_E"']
\end{tikzcd}
.$$

It follows that by (22), the following diagram commutes

$$\begin{tikzcd}
A \otimes_{\mathcal{D}} B \arrow[drr, "\underline{f \otimes_{Kl} g}"] \arrow[ddr, "\underline{f} \otimes_{\mathcal{D}} \underline{g}"'] \arrow[ddrrrr, bend left, "\underline{(f' \otimes_{Kl(T)} g') \circ (f \otimes_{Kl(T)} g)}"] \arrow[ddrrrr, bend right=60, "\underline{(f' \circ f) \otimes_{Kl(T)} (g' \circ g)}"'] & & & & \\
 & & T(X \otimes_{\mathcal{D}} Y) \arrow[r, "T(\underline{f' \otimes_{Kl} g'})"] & TT(C \otimes_{\mathcal{D}} E) \arrow[dr, "\mu_{C \otimes_{\mathcal{D}} E}"] & \\
 & TX \otimes_{\mathcal{D}} TY \arrow[dr, "T(\underline{f'}) \otimes_{\mathcal{D}} T(\underline{g'})"'] & & & T(C \otimes_{\mathcal{D}} E), \\
 & & TTC \otimes_{\mathcal{D}} TTE \arrow[r, "\mu_C \otimes_{\mathcal{D}} \mu_E"'] & TC \otimes_{\mathcal{D}} TE \arrow[ur, "\nabla_{C,E}"'] &
\end{tikzcd}$$

which confirms the last condition for the tensor product to be a bifunctor.

We will now denote $\otimes_{Kl(T)}$ by $\otimes$ for simplicity.

Recall that F is defined as follows.

- On objects : For every X in $Ob(\mathcal{D})$, $F(X) = X$.
- On morphisms : For every $f : X \to Y$ in $\mathcal{D}$, $F(f) : X \to Y$ is represented in $\mathcal{D}$ by

$$\underline{F(f)} := \eta_Y \circ f.$$

We have that F is a functor since for all object C in $\mathcal{D}$,

$$F(1_C) = \eta_C \circ 1_C = \eta_C,$$

which is the identity on C in $\mathcal{D}_{\mathbb{T}}$ and for all morphisms $f : A \to X$, $g : X \to Y$ in $\mathcal{D}$, if we denote new morphisms in $Kl(T)$ by their representation in $\mathcal{D}$: $\tilde{f} := \underline{F(f)} = \eta_X \circ f : A \to T(X)$ and $\tilde{g} := \underline{F(g)} = \eta_Y \circ g : X \to T(Y)$, then composition is preserved, as we now show. Observe that

$$\widetilde{g \circ f} = \eta_Y \circ g \circ f, \quad \tilde{f} \circ \tilde{g} = \mu_Y \circ T\eta_Y \circ Tg \circ \eta_X \circ f,$$

and

$$\begin{tikzcd}
 & TY \\
TY \arrow[ur, equal] \arrow[r, "T\eta_Y"'] & TTY, \arrow[u, "\mu_Y"']
\end{tikzcd}$$

commutes by the diagram (5) in the definition of a monad, while

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
TX \arrow[r, "Tg"'] & TY,
\end{tikzcd}$$

commutes by naturality of η. Putting that together, we have that the following big diagram commutes:

$$\begin{tikzcd}
 & & X \arrow[dr, "g"] \arrow[ddll, equal, bend left] & & \\
 & A \arrow[ur, "f"] \arrow[dl, "f"'] \arrow[r, phantom, "\circlearrowleft"] & & Y \arrow[dr, "\eta_Y"] & \\
X \arrow[dr, "\eta_X"'] \arrow[rrrr, phantom, "\circlearrowleft"] & & & & TY \\
 & TX \arrow[dr, "Tg"'] & & TTY \arrow[ur, "\mu_Y"'] \arrow[l, phantom, "\circlearrowleft"] & \\
 & & TY \arrow[ur, "T\eta_Y"'] \arrow[uurr, equal, bend left] & &
\end{tikzcd}$$

and so

$$\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}. \tag{23}$$

Then this functor satisfies the conditions we want, i.e.,

- *F is the identity on objects.*
- *The tensor product* is defined on objects by

$$A \otimes_{Kl} B = A \otimes_{\mathcal{D}} B$$

  for all A, B in $Kl(T)$. For any morphisms $f : A \to X$, $g : B \to Y$ in $\mathcal{D}$, the product is preserved:

$$\begin{aligned}
\underline{F(f \otimes_{\mathcal{D}} g)} &= \widetilde{f \otimes_{\mathcal{D}} g} = \eta_{X \otimes_{\mathcal{D}} Y} \circ (f \otimes_{\mathcal{D}} g) \\
&= \nabla_{X,Y} \circ (\eta_X \otimes_{\mathcal{D}} \eta_Y) \circ (f \otimes_{\mathcal{D}} g) = \quad = \nabla_{X,Y} \circ (\tilde{f} \otimes_{\mathcal{D}} \tilde{g}) \\
&= \underline{F(f) \otimes_{Kl} F(g)},
\end{aligned}$$

  in $\mathcal{D}$ by the diagram (8) from the definition 2.8 of a symmetric monoidal monad. It follows that in $Kl(T)$,

$$F(f \otimes_{\mathcal{D}} g) = F(f) \otimes_{Kl} F(g).$$

- *The images of the associator, the unitors and the braiding through F are natural regarding all the morphisms in $Kl(T)$.*
  - *$F(\alpha_{\mathcal{D}})$ is natural.* If $f : X \to A$, $g : Y \to B$ and $h : Z \to C$ are in $Kl(T)$ with representatives $\underline{f} : X \to TA$, $\underline{g} : Y \to TB$ and $\underline{h} : Z \to TC$ in $\mathcal{D}$, then $(f \otimes_{Kl(T)} g) \otimes_{Kl(T)} h$ and $f \otimes_{Kl(T)} (g \otimes_{Kl(T)} h)$ are represented in $\mathcal{D}$ by

$$\underline{(f \otimes_{Kl(T)} g) \otimes_{Kl(T)} h} = \nabla_{A \otimes B, C} \circ \underline{(f \otimes_{Kl(T)} g)} \otimes_{\mathcal{D}} \underline{h} = \nabla_{A \otimes B, C} \circ (\nabla_{A,B} \otimes_{\mathcal{D}} 1_{TC}) \circ (\underline{f} \otimes_{\mathcal{D}} \underline{g}) \otimes_{\mathcal{D}} \underline{h},$$

$$\underline{f \otimes_{Kl(T)} (g \otimes_{Kl(T)} h)} = \nabla_{A, B \otimes C} \circ \underline{f} \otimes_{\mathcal{D}} \underline{(g \otimes_{Kl(T)} h)} = \nabla_{A, B \otimes C} \circ (1_{TA} \otimes_{\mathcal{D}} \nabla_{C,D}) \circ \underline{f} \otimes_{\mathcal{D}} (\underline{g} \otimes_{\mathcal{D}} \underline{h})$$

    We want to prove that the following diagram commutes in $Kl(T)$,

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "(f \otimes_{Kl(T)} g) \otimes_{Kl(T)} h"] \arrow[d, "F(a_{X,Y,Z})"'] & (A \otimes B) \otimes C \arrow[d, "F(a_{A,B,C})"] \\
X \otimes (Y \otimes Z) \arrow[r, "f \otimes_{Kl(T)} (g \otimes_{Kl(T)} h)"] & A \otimes (B \otimes C),
\end{tikzcd}$$

    which is equivalent to the following diagram commuting in $\mathcal{D}$,

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "(\underline{f}\otimes_{\mathcal{D}}\underline{g})\otimes_{\mathcal{D}}\underline{h}"] \arrow[d, "a_{X,Y,Z}"'] & (TA\otimes TB)\otimes TC \arrow[r, "\nabla_{A,B}\otimes_{\mathcal{D}}1_{TC}"] & T(A\otimes B)\otimes TC \arrow[d, "\nabla_{A\otimes B,C}"] \\
X\otimes(Y\otimes Z) \arrow[d, "\eta_{X\otimes(Y\otimes Z)}"'] & & T((A\otimes B)\otimes C) \arrow[d, "T(a_{A,B,C})"] \\
T(X\otimes(Y\otimes Z)) \arrow[d, "T(\underline{f}\otimes_{\mathcal{D}}(\underline{g}\otimes_{\mathcal{D}}\underline{h}))"'] & & T(A\otimes(B\otimes C)) \arrow[d, "\eta_{T(A\otimes(B\otimes C))}"] \\
T(TA\otimes(TB\otimes TC)) \arrow[d, "T(1_{TA}\otimes_{\mathcal{D}}\nabla_{B,C})"'] & & TT(A\otimes(B\otimes C)) \arrow[d, "\mu_{A\otimes(B\otimes C)}"] \\
T(TA\otimes T(B\otimes C)) \arrow[r, "T(\nabla_{A,B\otimes C})"'] & TT(A\otimes(B\otimes C)) \arrow[r, "\mu_{A\otimes(B\otimes C)}"'] & T(A\otimes(B\otimes C)).
\end{tikzcd}$$

We have, by naturality of the associator in $\mathcal{D}$, that the following diagram commutes,

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "(\underline{f}\otimes_{\mathcal{D}}\underline{g})\otimes_{\mathcal{D}}\underline{h}"] \arrow[d, "a_{X,Y,Z}"'] & (TA\otimes TB)\otimes TC \arrow[d, "a_{TA,TB,TC}"] \\
X\otimes(Y\otimes Z) \arrow[r, "\underline{f}\otimes_{\mathcal{D}}(\underline{g}\otimes_{\mathcal{D}}\underline{h})"'] & TA\otimes(TB\otimes TC).
\end{tikzcd}$$

Moreover, T is a lax monoidal functor, so by diagram (1), the following diagram commutes,

$$\begin{tikzcd}
(TA\otimes TB)\otimes TC \arrow[r, "\nabla_{A,B}\otimes_{\mathcal{D}}1_{TC}"] \arrow[d, "a_{TA,TB,TC}"'] & T(A\otimes B)\otimes TC \arrow[d, "\nabla_{A\otimes B,C}"] \\
TA\otimes(TB\otimes TC) \arrow[d, "1_{TA}\otimes_{\mathcal{D}}\nabla_{B,C}"'] & T((A\otimes B)\otimes C) \arrow[d, "T(a_{A,B,C})"] \\
TA\otimes T(B\otimes C) \arrow[r, "\nabla_{A,B\otimes C}"'] & T(A\otimes(B\otimes C)).
\end{tikzcd}$$

By naturality of η, we have that the following diagram commutes

$$\begin{tikzcd}
X\otimes(Y\otimes Z) \arrow[r, "\underline{f}\otimes_{\mathcal{D}}(\underline{g}\otimes_{\mathcal{D}}\underline{h})"'] \arrow[d, "\eta_{X\otimes(Y\otimes Z)}"'] & TA\otimes(TB\otimes TC) \arrow[d, "1_{TA}\otimes_{\mathcal{D}}\nabla_{B,C}"'] & \\
T(X\otimes(Y\otimes Z)) \arrow[d, "T(\underline{f}\otimes_{\mathcal{D}}(\underline{g}\otimes_{\mathcal{D}}\underline{h}))"'] & TA\otimes T(B\otimes C) \arrow[r, "\nabla_{A,B\otimes C}"'] & T(A\otimes(B\otimes C)) \arrow[ddl, "\eta_{T(A\otimes(B\otimes C))}"'] \\
T(TA\otimes(TB\otimes TC)) \arrow[d, "T(1_{TA}\otimes_{\mathcal{D}}\nabla_{B,C})"'] & & \\
T(TA\otimes T(B\otimes C)) \arrow[r, "T(\nabla_{A,B\otimes C})"'] & TT(A\otimes(B\otimes C)). &
\end{tikzcd}$$

We have directly that the following diagram commutes

$$\begin{tikzcd}
 & T(A\otimes(B\otimes C)) \arrow[d, "\eta_{T(A\otimes(B\otimes C))}"] \arrow[ddl, "\eta_{T(A\otimes(B\otimes C))}"'] \\
 & TT(A\otimes(B\otimes C)) \arrow[d, "\mu_{A\otimes(B\otimes C)}"] \\
TT(A\otimes(B\otimes C)) \arrow[r, "\mu_{A\otimes(B\otimes C)}"'] & T(A\otimes(B\otimes C)).
\end{tikzcd}$$

Putting all the diagrams above together, we have that the big diagram below commutes,

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "(\underline{f} \otimes_{\mathcal{D}} \underline{g}) \otimes_{\mathcal{D}} \underline{h}"] \arrow[d, "a_{X,Y,Z}"'] & (TA \otimes TB) \otimes TC \arrow[r, "\nabla_{A,B} \otimes_{\mathcal{D}} 1_{TC}"] \arrow[d, "a_{TA,TB,TC}"'] & T(A \otimes B) \otimes TC \arrow[d, "\nabla_{A \otimes B, C}"] \\
X \otimes (Y \otimes Z) \arrow[r, "\underline{f} \otimes_{\mathcal{D}} (\underline{g} \otimes_{\mathcal{D}} \underline{h})"'] \arrow[d, "\eta_{X \otimes (Y \otimes Z)}"'] & TA \otimes (TB \otimes TC) \arrow[d, "1_{TA} \otimes_{\mathcal{D}} \nabla_{B,C}"'] & T((A \otimes B) \otimes C) \arrow[d, "T(a_{A,B,C})"] \\
T(X \otimes (Y \otimes Z)) \arrow[d, "T(\underline{f} \otimes_{\mathcal{D}} (\underline{g} \otimes_{\mathcal{D}} \underline{h}))"'] & TA \otimes T(B \otimes C) \arrow[r, "\nabla_{A, B \otimes C}"'] & T(A \otimes (B \otimes C)) \arrow[d, "\eta_{T(A \otimes (B \otimes C))}"] \arrow[ddl, "\eta_{T(A \otimes (B \otimes C))}"'] \\
T(TA \otimes (TB \otimes TC)) \arrow[d, "T(1_{TA} \otimes_{\mathcal{D}} \nabla_{B,C})"'] & & TT(A \otimes (B \otimes C)) \arrow[d, "\mu_{A \otimes (B \otimes C)}"] \\
T(TA \otimes T(B \otimes C)) \arrow[r, "T(\nabla_{A, B \otimes C})"'] & TT(A \otimes (B \otimes C)) \arrow[r, "\mu_{A \otimes (B \otimes C)}"'] & T(A \otimes (B \otimes C)),
\end{tikzcd}$$

confirming that $F(a)$ is natural.

– *$F(\lambda)$ and $F(\rho)$ are natural.* If $f : X \to Y$ is in $Kl(T)$, represented in $\mathcal{D}$ by $\underline{f} : X \to Y$, then $1_I \otimes_{Kl(T)} f : I \otimes X \to I \otimes Y$ and $f \otimes_{Kl(T)} 1_I : X \otimes I \to Y \otimes I$ are represented in $\mathcal{D}$ by

$$\underline{1_I \otimes_{Kl(T)} f} = \nabla_{I,Y} \circ (\eta_I \otimes_{\mathcal{D}} \underline{f}) : I \otimes X \to T(I \otimes Y),$$
$$\underline{f \otimes_{Kl(T)} 1_I} = \nabla_{Y,I} \circ (\underline{f} \otimes_{\mathcal{D}} \eta_I) : X \otimes I \to T(Y \otimes I).$$

We want to prove that the following diagrams commute in $Kl(T)$,

$$\begin{tikzcd}
I \otimes X \arrow[r, "F(\lambda_X)"] \arrow[d, "1_I \otimes_{Kl(T)} f"'] & X \arrow[d, "f"] \\
I \otimes Y \arrow[r, "F(\lambda_Y)"'] & Y
\end{tikzcd}, \qquad
\begin{tikzcd}
X \otimes I \arrow[r, "F(\rho_X)"] \arrow[d, "f \otimes_{Kl(T)} 1_I"'] & X \arrow[d, "f"] \\
Y \otimes I \arrow[r, "F(\rho_Y)"'] & Y
\end{tikzcd}$$

which is equivalent to the following diagrams commuting in $\mathcal{D}$,

$$\begin{tikzcd}
I \otimes X \arrow[rr, "\lambda_X"] \arrow[d, "\eta_I \otimes_{\mathcal{D}} \underline{f}"'] & & X \arrow[r, "\eta_X"] & TX \arrow[d, "T\underline{f}"] \\
TI \otimes TY \arrow[d, "\nabla_{I,Y}"'] & & & TTY \arrow[d, "\mu_Y"] \\
T(I \otimes Y) \arrow[r, "T(\lambda_Y)"'] & TY \arrow[r, "T\eta_Y"'] & TTY \arrow[r, "\mu_Y"'] & TY,
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes I \arrow[rr, "\rho_X"] \arrow[d, "\underline{f} \otimes_{\mathcal{D}} \eta_I"'] & & X \arrow[r, "\eta_X"] & TX \arrow[d, "T\underline{f}"] \\
TY \otimes TI \arrow[d, "\nabla_{Y,I}"'] & & & TTY \arrow[d, "\mu_Y"] \\
T(Y \otimes I) \arrow[r, "T(\rho_Y)"'] & TY \arrow[r, "T\eta_Y"'] & TTY \arrow[r, "\mu_Y"'] & TY.
\end{tikzcd}$$

By bifunctoriality of $- \otimes_{\mathcal{D}} -$, the following diagrams commute,

$$\begin{tikzcd}
I \otimes X \arrow[d, "\eta_I \otimes_{\mathcal{D}} \underline{f}"'] \arrow[dr, "1_I \otimes_{\mathcal{D}} \underline{f}"] & \\
TI \otimes TY & I \otimes TY \arrow[l, "\eta_I \otimes_{\mathcal{D}} 1_{TY}"]
\end{tikzcd}, \qquad
\begin{tikzcd}
X \otimes I \arrow[d, "\underline{f} \otimes_{\mathcal{D}} \eta_I"'] \arrow[dr, "\underline{f} \otimes_{\mathcal{D}} 1_I"] & \\
TY \otimes TI & TY \otimes I \arrow[l, "1_{TY} \otimes_{\mathcal{D}} \eta_I"]
\end{tikzcd}.$$

By naturality of λ and ρ, the following diagrams commute,

$$\begin{tikzcd}
I\otimes X \arrow[rr, "\lambda_X"] \arrow[dr, "1_I\otimes_{\mathcal{D}}\underline{f}"'] & & X, \arrow[dl, "\underline{f}"] \\
& I\otimes TY \arrow[d, "\lambda_{TY}"'] & \\
& TY &
\end{tikzcd}
\qquad
\begin{tikzcd}
X\otimes I \arrow[rr, "\rho_X"] \arrow[dr, "\underline{f}\otimes_{\mathcal{D}}1_I"'] & & X. \arrow[dl, "\underline{f}"] \\
& I\otimes TY \arrow[d, "\lambda_{TY}"'] & \\
& TY &
\end{tikzcd}$$

Since T is a lax symmetric monoidal functor, the following diagrams commute,

$$\begin{tikzcd}
TI\otimes TY \arrow[d, "\nabla_{I,Y}"'] & I\otimes TY \arrow[l, "\eta_I\otimes_{\mathcal{D}}1_{TY}"] \arrow[d, "\lambda_{TY}"'] \\
T(I\otimes Y) \arrow[r, "T(\lambda_Y)"'] & TY,
\end{tikzcd}
\quad
\begin{tikzcd}
TY\otimes TI \arrow[d, "\nabla_{Y,I}"'] & TY\otimes I \arrow[l, "1_{TY}\otimes_{\mathcal{D}}\eta_I"] \arrow[d, "\rho_{TY}"'] \\
T(Y\otimes I) \arrow[r, "T(\rho_Y)"'] & TY.
\end{tikzcd}$$

By diagram (5) in Definition of a monad,

$$\mu_Y \circ T\eta_Y = \mu_Y \circ \eta_{TY},$$

and by naturality of η, the following diagrams commute:

$$\begin{tikzcd}
& X \arrow[r, "\eta_X"] \arrow[dl, "\underline{f}"] & TX \arrow[d, "T\underline{f}"] \\
& & TTY \arrow[d, "\mu_Y"] \\
TY \arrow[r, "\eta_{TY}"] \arrow[rr, bend right, "\mu_Y\circ T\eta_Y"'] & TTY \arrow[r, "\mu_Y"] \arrow[ur, equal] & TY
\end{tikzcd},
\qquad
\begin{tikzcd}
& X \arrow[r, "\eta_X"] \arrow[dl, "\underline{f}"] & TX \arrow[d, "T\underline{f}"] \\
& & TTY \arrow[d, "\mu_Y"] \\
TY \arrow[r, "\eta_{TY}"] \arrow[rr, bend right, "\mu_Y\circ T\eta_Y"'] & TTY \arrow[r, "\mu_Y"] \arrow[ur, equal] & TY
\end{tikzcd}.$$

Putting all the diagrams above together, we have that the following two big diagrams commute

$$\begin{tikzcd}
I\otimes X \arrow[rr, "\lambda_X"] \arrow[d, "\eta_I\otimes_{\mathcal{D}}\underline{f}"'] \arrow[dr, "1_I\otimes_{\mathcal{D}}\underline{f}"] & & X \arrow[r, "\eta_X"] \arrow[ddl, "\underline{f}"] & TX \arrow[d, "T\underline{f}"] \\
TI\otimes TY \arrow[d, "\nabla_{I,Y}"'] & I\otimes TY \arrow[l, "\eta_I\otimes_{\mathcal{D}}1_{TY}"] \arrow[d, "\lambda_{TY}"'] & & TTY \arrow[d, "\mu_Y"] \\
T(I\otimes Y) \arrow[r, "T(\lambda_Y)"'] & TY \arrow[r, "\eta_{TY}"] \arrow[rr, bend right, "\mu_Y\circ T\eta_Y"'] & TTY \arrow[r, "\mu_Y"] \arrow[ur, equal] & TY,
\end{tikzcd}$$

$$\begin{tikzcd}
X\otimes I \arrow[rr, "\rho_X"] \arrow[d, "\underline{f}\otimes_{\mathcal{D}}\eta_I"'] \arrow[dr, "\underline{f}\otimes_{\mathcal{D}}1_I"] & & X \arrow[r, "\eta_X"] \arrow[ddl, "\underline{f}"] & TX \arrow[d, "T\underline{f}"] \\
TY\otimes TI \arrow[d, "\nabla_{Y,I}"'] & TY\otimes I \arrow[l, "1_{TY}\otimes_{\mathcal{D}}\eta_I"] \arrow[d, "\rho_{TY}"'] & & TTY \arrow[d, "\mu_Y"] \\
T(Y\otimes I) \arrow[r, "T(\rho_Y)"'] & TY \arrow[r, "\eta_{TY}"] \arrow[rr, bend right, "\mu_Y\circ T\eta_Y"'] & TTY \arrow[r, "\mu_Y"] \arrow[ur, equal] & TY,
\end{tikzcd}$$

establishing that $F(\lambda)$ and $F(\rho)$ are natural in $Kl(T)$.

– *$F(B)$ is natural.* If $f : X \to A$ and $g : Y \to C$ are in $Kl(T)$, represented in $\mathcal{D}$ by $\underline{f} : X \to TA$ and $\underline{g} : Y \to TC$, then $f \otimes_{Kl(T)} g : X \otimes Y \to A \otimes C$ and $g \otimes_{Kl(T)} f : Y \otimes X \to C \otimes A$ are represented in $\mathcal{D}$ by

$$\underline{f \otimes_{Kl(T)} g} = \nabla_{X,Y} \circ (\underline{f} \otimes_{\mathcal{D}} \underline{g}) : X \otimes Y \to A \otimes C,$$

$$\underline{g \otimes_{Kl(T)} f} = \nabla_{X,Y} \circ (\underline{g} \otimes_{\mathcal{D}} \underline{f}) : Y \otimes X \to C \otimes A.$$

We want to prove that the following diagram commutes in $Kl(T)$,

$$\begin{tikzcd}
X \otimes Y \arrow[r, "B_{X,Y}"] \arrow[d, "f \otimes_{Kl(T)} g"'] & Y \otimes X \arrow[d, "g \otimes_{Kl(T)} f"] \\
A \otimes C \arrow[r, "B_{A,C}"'] & C \otimes A,
\end{tikzcd}$$

which is equivalent to the following diagram commuting in $\mathcal{D}$,

$$\begin{tikzcd}
X \otimes Y \arrow[r, "B_{X,Y}"] \arrow[d, "\underline{f} \otimes_{\mathcal{D}} \underline{g}"'] & Y \otimes X \arrow[r, "\eta_{Y \otimes X}"] & T(Y \otimes X) \arrow[d, "T(\underline{g} \otimes_{\mathcal{D}} \underline{f})"] \\
TA \otimes TC \arrow[d, "\nabla_{A,C}"'] & & T(TC \otimes TA) \arrow[d, "T\nabla_{C,A}"] \\
T(A \otimes C) \arrow[d, "T(B_{A,C})"'] & & TT(C \otimes A) \arrow[d, "\mu_{C \otimes A}"] \\
T(C \otimes A) \arrow[r, "T(\eta_{C \otimes A})"'] & TT(C \otimes A) \arrow[r, "\mu_{C \otimes A}"'] & T(C \otimes A).
\end{tikzcd}$$

By naturality of the braiding B and η, we have that the following diagrams commute,

$$\begin{tikzcd}
X \otimes Y \arrow[r, "B_{X,Y}"] \arrow[d, "\underline{f} \otimes_{\mathcal{D}} \underline{g}"'] & Y \otimes X \arrow[r, "\eta_{Y \otimes X}"] \arrow[d, "\underline{g} \otimes_{\mathcal{D}} \underline{f}"] & T(Y \otimes X) \arrow[d, "T(\underline{g} \otimes_{\mathcal{D}} \underline{f})"] \\
TA \otimes TC \arrow[r, "B_{TA,TC}"'] & TC \otimes TA \arrow[r, "\eta_{TC \otimes TA}"'] & T(TC \otimes TA),
\end{tikzcd}$$

$$\begin{tikzcd}
TC \otimes TA \arrow[r, "\eta_{TC \otimes TA}"] \arrow[d, "\nabla_{C,A}"] & T(TC \otimes TA) \arrow[d, "T\nabla_{C,A}"] \\
T(C \otimes A) \arrow[r, "\eta_{T(C \otimes A)}"'] & TT(C \otimes A).
\end{tikzcd}$$

Since T is lax symmetric monoidal, by Diagram (4), the following diagram commutes

$$\begin{tikzcd}
TA \otimes TC \arrow[r, "B_{TA,TC}"] \arrow[d, "\nabla_{A,C}"'] & TC \otimes TA \arrow[ddl, "\nabla_{C,A}"] \\
T(A \otimes C) \arrow[d, "T(B_{A,C})"'] & \\
T(C \otimes A). &
\end{tikzcd}$$

By definition of a monad, diagram (6), the following diagram commutes,

$$\begin{tikzcd}
 & T(C \otimes A) \arrow[r, "\eta_{T(C \otimes A)}"] \arrow[dl, equal] \arrow[dr, equal] & TT(C \otimes A) \arrow[d, "\mu_{C \otimes A}"] \\
T(C \otimes A) \arrow[r, "T(\eta_{C \otimes A})"'] & TT(C \otimes A) \arrow[r, "\mu_{C \otimes A}"'] & T(C \otimes A).
\end{tikzcd}$$

Putting all of the diagrams above together, we have that the following big diagram commutes,

$$\begin{tikzcd}
X \otimes Y \arrow[r, "B_{X,Y}"] \arrow[d, "\underline{f} \otimes_{\mathcal{D}} \underline{g}"'] & Y \otimes X \arrow[r, "\eta_{Y \otimes X}"] \arrow[d, "\underline{g} \otimes_{\mathcal{D}} \underline{f}"] & T(Y \otimes X) \arrow[d, "T(\underline{g} \otimes_{\mathcal{D}} \underline{f})"] \\
TA \otimes TC \arrow[r, "B_{TA,TC}"] \arrow[d, "\nabla_{A,C}"'] & TC \otimes TA \arrow[r, "\eta_{TC \otimes TA}"] \arrow[d, "\nabla_{C,A}"] \arrow[ddl, "\nabla_{C,A}"'] & T(TC \otimes TA) \arrow[d, "T\nabla_{C,A}"] \\
T(A \otimes C) \arrow[d, "T(B_{A,C})"'] & T(C \otimes A) \arrow[r, "\eta_{T(C \otimes A)}"] \arrow[dl, equal] \arrow[dr, equal] & TT(C \otimes A) \arrow[d, "\mu_{C \otimes A}"] \\
T(C \otimes A) \arrow[r, "T(\eta_{C \otimes A})"'] & TT(C \otimes A) \arrow[r, "\mu_{C \otimes A}"'] & T(C \otimes A),
\end{tikzcd}$$

establishing the naturality of $F(B)$.

So Lemma 2.20 applies, implying that $Kl(T)$ is a symmetric monoidal category, and that F is a strict symmetric monoidal functor. This yields the Proposition. ◻

Corollary 2.25. *[2, 3.2]. Let $\mathcal{D}$ be a Markov category and T a symmetric monoidal affine monad on $\mathcal{D}$. Then the Kleisli category $Kl(T)$ is again a Markov category in a canonical way.*

Proof. We want to apply the second part of Lemma 2.20. For the naturality of $F(del)$, we have to show that for every $f : X \to Y$ in $Kl(T)$,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "F(del_X)"'] & Y \arrow[d, "F(del_Y)"] \\
I \arrow[r, equal] & I,
\end{tikzcd}$$

commutes in $Kl(T)$. This means that in $\mathcal{D}$, the following diagram must commute

$$\begin{tikzcd}
X \arrow[r, "\underline{f}"] \arrow[dd, "\widetilde{del}_{\mathcal{D}_X}"'] & TY \arrow[d, "T(\widetilde{del}_{\mathcal{D}_Y})"] \\
 & TTI \arrow[d, "\mu_I"] \\
TI \arrow[r, equal] & TI.
\end{tikzcd}$$

Since $\mathcal{D}$ is a Markov category, for every $\underline{f} : X \to TY$ in $\mathcal{D}$,

$$\begin{tikzcd}
X \arrow[r, "\underline{f}"] \arrow[d, "del_{\mathcal{D}_X}"'] & TY, \arrow[dl, "del_{\mathcal{D}_{TY}}"] \\
I &
\end{tikzcd}$$

commutes and

$$\begin{tikzcd}
TI \arrow[r, "T(\eta_I)"] \arrow[d, equal] & TTI, \arrow[dl, "\mu_I"] \\
TI &
\end{tikzcd}$$

commutes by diagram (5) in Definition 2.4 of a monad. The diagram

$$\begin{tikzcd}
TY \arrow[r, "T(del_{\mathcal{D}_Y})"] \arrow[d, "del_{\mathcal{D}_{TY}}"'] & TI \arrow[d, "del_{\mathcal{D}_{TI}}"'] \\
I \arrow[r, equal] & I \arrow[u, bend right, "\eta_I"']
\end{tikzcd}$$

commutes by naturality of del and the fact that T is affine (i.e., $\eta_I : I \to TI$ is the inverse of the unique morphism $del_{TI} : TI \to I$) , so that the diagram

$$\begin{tikzcd}
 & TY \arrow[r, "T(del_{\mathcal{D}_Y})"] \arrow[dl, "del_{\mathcal{D}_{TY}}"'] & TI \arrow[dr, equal] & \\
I \arrow[rrr, "\eta_I"'] & & & TI,
\end{tikzcd}$$

commutes. Putting that together, we have that the following big diagram commutes

$$\begin{tikzcd}
X \arrow[rr, "\underline{f}"] \arrow[dr, "del_{\mathcal{D}_X}"'] \arrow[drrrrr, bend right=60, "\widetilde{del}_{\mathcal{D}_X}"'] & \arrow[d, phantom, "\circlearrowleft"] & TY \arrow[rr, "T(del_{\mathcal{D}_Y})"] \arrow[dl, "del_{\mathcal{D}_{TY}}"] \arrow[rrrr, bend left=40, "T(\widetilde{del}_{\mathcal{D}_Y})"] & \arrow[d, phantom, "\circlearrowleft"] & TI \arrow[rr, "T(\eta_I)"] \arrow[dr, equal] & \arrow[d, phantom, "\circlearrowleft"] & TTI. \arrow[dl, "\mu_I"] \\
 & I \arrow[rrrr, "\eta_I"'] & & & & TI &
\end{tikzcd}$$

So $Kl(T)$ is a Markov category. $\square$

Example 2.26. In 2.15, we have defined the Kleisli category *MultiVar* of a symmetric monoidal affine monad on the Markov category *FinSet*. Then we have the *Multivar* is also a Markov category.

Example 2.27. We would like to see if *FinStoch* is the Kleisli category of a monad on *FinSet*. For $f : X \to Y$ in *FinStoch*, we want to interpret it as a function from X to TY, a finite set. But if we interpret f as a function, then $f(x) = (\lambda_1, \lambda_2, ...\lambda_n)$, with $n = |Y|$ and $\sum_{0<i\leq n} \lambda_i = 1$, so we need $f(x)$ to belong to an infinite space (not even countable), so TY cannot be in *FinSet*.

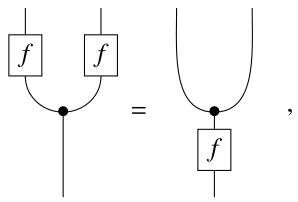
3 Properties of Markov categories

In this section, we will see some properties that all the Markov categories have, some particular morphisms, and some interesting properties that a Markov category can have (but not necessarily).

3.1 Deterministic morphisms

We will first study some particular morphisms in Markov categories, that can be interpreted as functions or *deterministic morphisms*.

Definition 3.1. [2, 10.1]. A morphism $f : X \to Y$ in a Markov category is *deterministic* if it respects the comultiplication,



i.e., f is a morphism of comonoids and the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "copy_X"'] & Y \arrow[d, "copy_Y"] \\
X \otimes X \arrow[r, "f \otimes f"'] & Y \otimes Y.
\end{tikzcd}$$

Example 3.2. In *Set*,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\
X \times X \arrow[r, "f \times f"'] & Y \times Y,
\end{tikzcd}$$

commutes for every function $f : X \to Y$. So every morphism in *Set* is deterministic.

Example 3.3. In *FinStoch*, if $f : X \to Y$ is deterministic,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "copy_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "copy_Y"] \\ X \times X \arrow[r, "f\otimes f"'] & Y \times Y. \end{tikzcd}$$

For every $(y_1, y_2) \in Y \times Y$ and $x_0 \in X$, on the one hand,

$$X \xrightarrow{copy_X} X \times X \xrightarrow{f\otimes f} Y \times Y,$$

gives

$$\begin{aligned} f \otimes f \circ copy_X(y_1, y_2|x_0) &= \sum_{(x_1,x_2)\in X\times X} f \otimes f(y_1, y_2|x_1, x_2) \cdot copy_X(x_1, x_2|x_0) \\ &= f \otimes f(y_1, y_2|x_0, x_0) \\ &= f(y_1|x_0) \cdot f(y_2|x_0). \end{aligned}$$

On the other hand,

$$X \xrightarrow{f} Y \xrightarrow{copy_Y} Y \times Y,$$

gives

$$\begin{aligned} copy_Y \circ f(y_1, y_2|x_0) &= \sum_{y_0\in Y} copy_Y(y_1, y_2|y_0) \cdot f(y_0|x_0) \\ &= f(y_1|x_0) \cdot \delta_{y_1,y_2}, \end{aligned}$$

so that f deterministic implies that for every $(y_1, y_2) \in Y \times Y$ and $x_0 \in X$,

$$f(y_1|x_0) \cdot f(y_2|x_0) = f(y_1|x_0) \cdot \delta_{y_1,y_2} = \begin{cases} f(y_1|x_0) & \text{if } y_1 = y_2 \\ 0 & \text{otherwise,} \end{cases}$$

which happens if and only if

$$f(y_1|x_0) = 0,$$

or

$$f(y_2|x_0) = \begin{cases} f(y_1|x_0) = 1 & \text{if } y_1 = y_2 \\ 0 & \text{if } y_1 \neq y_2. \end{cases}$$

Since

$$\sum_{y\in Y} f(y|x_0) = 1,$$

for every x_0, there exists a unique $y \in Y$ such that $f(y|x_0) = 1$, and $f(y_2|x_0) = \delta_{y,y_2}$ for every $y_2 \in Y$. This is consistent with the probabilistic definition of a deterministic function : for every $x_0 \in X$, we are sure to assign only one value $y \in Y$, so that $f(y_0|x_0) = \delta_{y,y_0}$. The deterministic morphisms also coincide with the images of functions $f : X \to Y$ in *FinSet* through the standard inclusion functor $FinSet \to FinStoch$ defined in Exemple 2.22.

Example 3.4. In *Gauss*, a morphism $f : n \to m$ defined by (M, C, s), $M \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{m\times m}$, $s \in \mathbb{R}^m$, is deterministic if

$$\begin{tikzcd} n \arrow[r, "copy_n"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & 2n \arrow[d, "f\otimes f"] \\ m \arrow[r, "copy_m"'] & 2m. \end{tikzcd}$$

On the one hand,

$$n \xrightarrow{copy_n} 2n \xrightarrow{f\otimes f} 2m,$$

gives

$$(f \otimes f) \circ copy_n = (M \oplus M, C \oplus C, s \oplus s) \circ \left(\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, 0, 0\right)$$
$$= \left((M \oplus M)\begin{pmatrix} 1_n \\ 1_n \end{pmatrix}, (M \oplus M) \cdot 0 \cdot (M \oplus M)^T + C \oplus C, (M \oplus M) \cdot 0 + s \oplus s\right)$$
$$= \left(\begin{pmatrix} M \\ M \end{pmatrix}, C \oplus C, s \oplus s\right).$$

On the other hand,

$$n \xrightarrow{f} m \xrightarrow{copy_m} 2m,$$

gives

$$copy_m \circ f = \left(\begin{pmatrix} 1_m \\ 1_m \end{pmatrix}, 0, 0\right) \circ (M, C, s)$$
$$= \left(\begin{pmatrix} 1_m \\ 1_m \end{pmatrix} M, \begin{pmatrix} 1_m \\ 1_m \end{pmatrix} C \begin{pmatrix} 1_m \\ 1_m \end{pmatrix}^T + 0, \begin{pmatrix} 1_m \\ 1_m \end{pmatrix} s + 0\right)$$
$$= \left(\begin{pmatrix} M \\ M \end{pmatrix}, \begin{pmatrix} C & C \\ C & C \end{pmatrix}, \begin{pmatrix} s \\ s \end{pmatrix}\right).$$

For every $M \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{m \times m}$, $s \in \mathbb{R}^m$,

$$\begin{pmatrix} M \\ M \end{pmatrix} = \begin{pmatrix} M \\ M \end{pmatrix}$$
$$C \oplus C = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$$
$$s \oplus s = \begin{pmatrix} s \\ s \end{pmatrix},$$

so that f is deterministic if and only if

$$\begin{pmatrix} C & C \\ C & C \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix},$$

i.e., $C = 0$. This corresponds to the probabilistic definition of a deterministic morphism since $C = 0$ corresponds to morphism with no covariance, i.e., deterministic variables.

Definition 3.5. Let $\mathcal{D}$ be a category. Then a morphism $f : X \to Y$ in $\mathcal{D}$ is an *epimorphism* if, for all object Z and all morphisms $g_1, g_2 : Y \to Z$ in $\mathcal{D}$,

$$g_1 \circ f = g_2 \circ f,$$

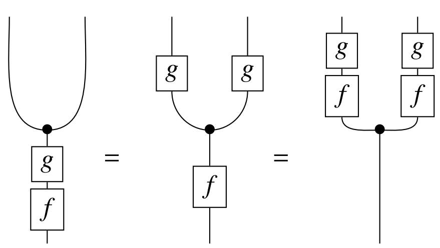
implies

$$g_1 = g_2.$$

Lemma 3.6. *[2, 10.9]. Let C be a Markov category and let $f : X \to Y$ and $g : Y \to Z$ be morphisms in C. Then:*

1. *If f and g are deterministic, then so is gf.*
2. *If gf is deterministic and f is a deterministic epimorphism, then g is deterministic as well.*
3. *If f and g are inverse isomorphisms and if one of them is deterministic, then so is the other.*

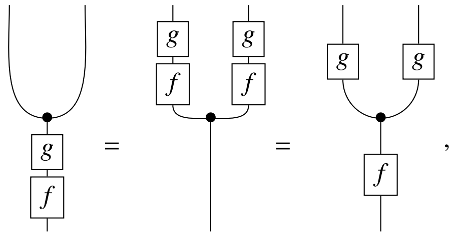
Proof. 1. Suppose f and g are deterministic. Then,



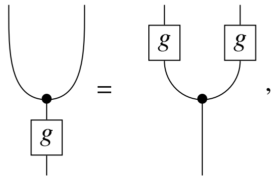
,

where the first equality comes from the fact that g is deterministic, and the second from the fact that f is deterministic.

2. Suppose that gf is deterministic and that f is a deterministic epimorphism. Then,

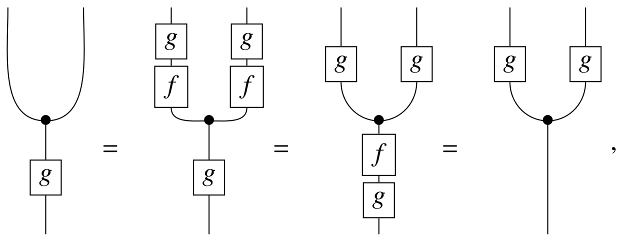


where the first equality holds since gf is deterministic, and since f is an epimorphism, we get that



i.e., g is deterministic.

3. Suppose that f and g are inverse isomorphisms. We can suppose, without lost of generality, that f is deterministic. Then,



where the first and last equalities hold since f and g are inverse isomorphisms, and the second holds since f is deterministic. We then have that g is deterministic.

□

With the result above in hand, we can construct a subcategory made of the deterministic morphisms:

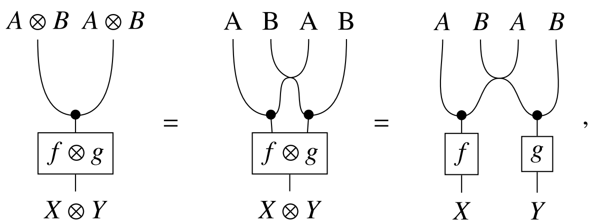
Definition 3.7. C_{det} is the category defined as follows.

- The objects of C_{det} are exactly the objects of C
- The morphisms of C_{det} are the deterministic morphisms in C.

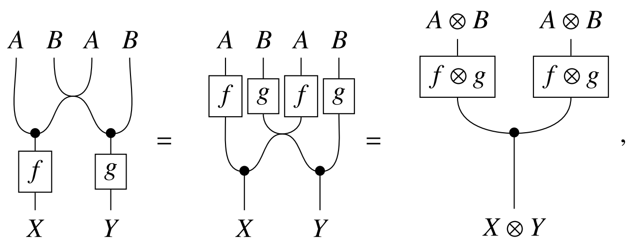
We can actually construc a Markov subcategory out of these deterministic morphisms.

Lemma 3.8. *[2, 10.12]. The subcategory of deterministic morphisms $C_{det} \subseteq C$ is closed under tensor product and contains all structure maps, meaning the associators, unitors, del, copy, and swap.*

Proof. • *C_{det} is closed under tensor product.* Since $ObC_{det} = Ob_C$, for all X, Y objects in C_{det}, $X \otimes Y$ is also an object in C_{det}. Now suppose $f : X \to A$, $g : Y \to B$ are deterministic morphisms. Then, their tensor product is also deterministic:

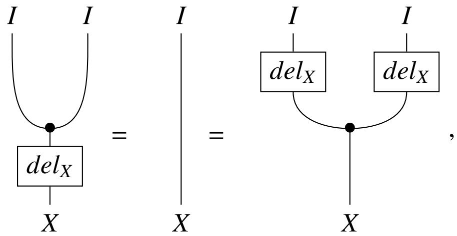


where the equalities hold by (14).

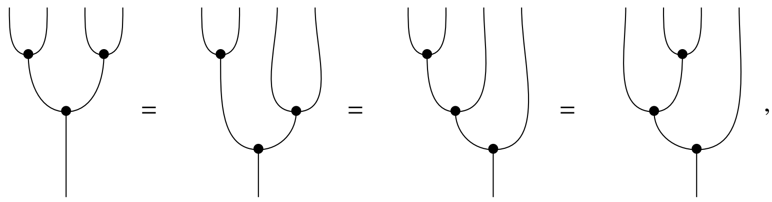


where the first equality holds by the fact that f and g are deterministic, and the second by (14). This yields that C_{det} is closed under tensor product.

- *C_{det} contains all structure maps.*
  - for every object X in C, del_X is deterministic:

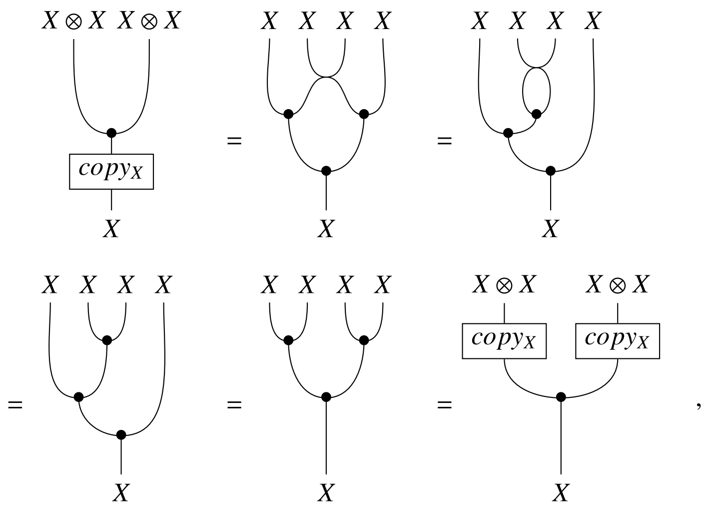


    by counitality (12).
  - for every object X in C, $copy_X$ is deterministic:



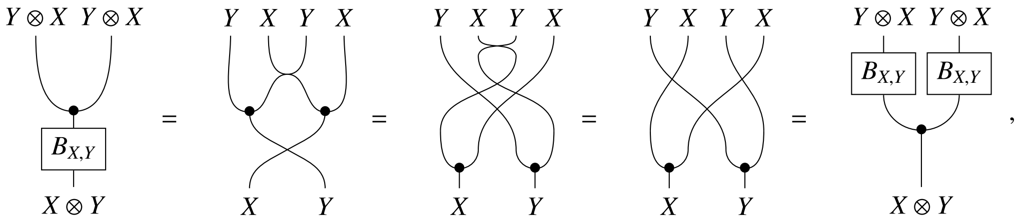
(24)

    where for the second and third equality, we used coassociativity of the objects in C (11). And this yields that, for all object X in C:



    where in the third equality, we used cocommutativity of the objects in C (13)

– for every pair of objects X, Y in C, $B_{X,Y}$ is deterministic:



where the first and fourth equalities hold by (14), the second equality by naturality of the braiding:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "copy_X \otimes copy_Y"] \arrow[d, "B_{X,Y}"'] \arrow[dr, phantom, "\circlearrowleft"] & X \otimes X \otimes Y \otimes Y \arrow[d, "B_{X\otimes X, Y\otimes Y}"] \\
Y \otimes X \arrow[r, "copy_Y \otimes copy_X"'] & Y \otimes Y \otimes X \otimes X,
\end{tikzcd}$$

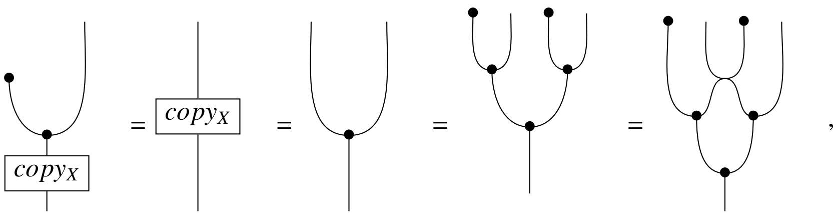
and the third equality holds by symmetry of the braiding and first and second hexagon identities:

$$\begin{aligned}
&(1_Y \otimes B_{Y,X} \otimes 1_X) \circ B_{X\otimes X, Y\otimes Y} \\
&= (1_Y \otimes B_{X\otimes X,Y}) \circ (B_{X\otimes X,Y} \otimes 1_Y) \\
&= (1_Y \otimes B_{Y,X} \otimes 1_X) \circ (1_Y \otimes (B_{X,Y} \otimes 1_X \circ 1_X \otimes B_{X,Y})) \circ ((B_{X,Y} \otimes 1_X \circ 1_X \otimes B_{X,Y}) \otimes 1_Y) \\
&= (1_Y \otimes B_{Y,X} \otimes 1_X) \circ (1_Y \otimes B_{X,Y} \otimes 1_X) \circ (1_Y \otimes 1_X \otimes B_{X,Y}) \circ (B_{X,Y} \otimes 1_X \otimes 1_Y) \circ (1_X \otimes B_{X,Y} \otimes 1_Y) \\
&= (B_{X,Y} \otimes B_{X,Y}) \circ (1_X \otimes B_{X,Y} \otimes 1_Y).
\end{aligned}$$

– for every object X in C, λ_X and ρ_X are deterministic: by Lemma 3.6, we can prove that λ_X^{-1} and ρ_X^{-1}. By counitality (10) of the objects in C,

$$\lambda_X^{-1} = (del_X \otimes 1_X) \circ copy_X,$$
$$\rho_X^{-1} = (1_X \otimes del_X) \circ copy_X,$$

And we have,



where the first and third equalities hold by counitality (12), and the fourth by (24). So we have that λ_X^{-1} is deterministic, and similarly for ρ_X^{-1}.

– for every triple of objects X, Y, Z in C, the associator $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$ is deterministic. We will detail the commutativity of the diagram introduced by T. Fritz in [2] 10.12. The diagram

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[rr, "copy_{(X\otimes Y)\otimes Z}"] \arrow[d, equal] & & ((X \otimes Y) \otimes Z) \otimes ((X \otimes Y) \otimes Z) \arrow[d, "gr_{(X\otimes Y)\otimes Z}"] \\
(X \otimes Y) \otimes Z \arrow[r, "copy_{X\otimes Y}\otimes 1"'] & ((X \otimes Y) \otimes (X \otimes Y)) \otimes Z \arrow[r, "1\otimes copy_Z"'] & ((X \otimes Y) \otimes (X \otimes Y)) \otimes (Z \otimes Z),
\end{tikzcd}$$

commutes by (15). The diagram

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[rr, "copy_{X\otimes Y}\otimes 1"] \arrow[d, equal] & & ((X \otimes Y) \otimes (X \otimes Y)) \otimes Z \arrow[d, "gr_{X,Y}\otimes 1"] \\
(X \otimes Y) \otimes Z \arrow[r, "(1\otimes copy_Y)\otimes 1"'] & (X \otimes (Y \otimes Y)) \otimes Z \arrow[r, "(copy_X\otimes 1)\otimes 1"'] & ((X \otimes X) \otimes (Y \otimes Y)) \otimes Z,
\end{tikzcd}$$

commutes by (15) and by the fact that $(copy_X \otimes 1_{Y\otimes Y}) \circ (1_X \otimes copy_Y) = copy_X \otimes copy_Y = (1_{X\otimes X} \otimes copy_Y) \circ (copy_X \otimes 1_Y)$ by functoriality of $- \otimes -$. The diagram

$$\begin{tikzcd}
((X \otimes Y) \otimes (X \otimes Y)) \otimes Z \arrow[r, "1\otimes copy_Z"] \arrow[d, "gr_{X,Y}\otimes 1"'] & ((X \otimes Y) \otimes (X \otimes Y)) \otimes (Z \otimes Z) \arrow[d, "gr_{X,Y}\otimes 1"] \\
((X \otimes X) \otimes (Y \otimes Y)) \otimes Z \arrow[r, "1\otimes copy_Z"'] & ((X \otimes X) \otimes (Y \otimes Y)) \otimes (Z \otimes Z),
\end{tikzcd}$$

commutes by functoriality of $- \otimes -$. The following three diagrams commute by naturality of the associator.

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "1\otimes copy_Y)\otimes 1"] \arrow[d, "a_{X,Y,Z}"', "\cong"] & (X \otimes (Y \otimes Y)) \otimes Z \arrow[d, "\cong"', "a_{X,Y\otimes Y,Z}"] \\
X \otimes (Y \otimes Z) \arrow[r, "1\otimes(copy_Y\otimes 1_Y)"'] & X \otimes ((Y \otimes Y) \otimes Z),
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes (Y \otimes Y)) \otimes Z \arrow[r, "(copy_X\otimes 1)\otimes 1"] \arrow[d, "a_{X,Y\otimes Y,Z}"', "\cong"] & ((X \otimes X) \otimes (Y \otimes Y)) \otimes Z \arrow[d, "\cong"', "a_{(X\otimes X,(Y\otimes Y),Z}"] \\
X \otimes ((Y \otimes Y) \otimes Z) \arrow[r, "copy_X\otimes 1"'] & (X \otimes X) \otimes ((Y \otimes Y) \otimes Z),
\end{tikzcd}$$

$$\begin{tikzcd}
((X \otimes X) \otimes (Y \otimes Y)) \otimes Z \arrow[r, "1\otimes copy_Z"] \arrow[d, "a_{X\otimes X,(Y\otimes Y),Z}"', "\cong"] & ((X \otimes X) \otimes (Y \otimes Y)) \otimes (Z \otimes Z) \arrow[d, "\cong"', "a_{X\otimes X,Y\otimes Y,Z\otimes Z}"] \\
(X \otimes X) \otimes ((Y \otimes Y) \otimes Z) \arrow[r, "1\otimes(1\otimes copy_Z)"'] & (X \otimes X) \otimes ((Y \otimes Y) \otimes (Z \otimes Z)).
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
X \otimes ((Y \otimes Y) \otimes Z) \arrow[r, "copy_X\otimes 1"] \arrow[d, equal] & (X \otimes X) \otimes ((Y \otimes Y) \otimes Z) \arrow[r, "1\otimes(1\otimes copy_Z)"] & (X \otimes X) \otimes ((Y \otimes Y) \otimes (Z \otimes Z)) \arrow[d, equal] \\
X \otimes ((Y \otimes Y) \otimes Z) \arrow[r, "1\otimes(1\otimes copy_Z)"'] & X \otimes ((Y \otimes Y) \otimes (Z \otimes Z)) \arrow[r, "copy_X\otimes 1"'] & (X \otimes X) \otimes ((Y \otimes Y) \otimes (Z \otimes Z)),
\end{tikzcd}$$

commutes by functoriality of $- \otimes -$. The diagram

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "1\otimes(copy_Y\otimes 1)"] \arrow[d, equal] & X \otimes ((Y \otimes Y) \otimes Z) \arrow[r, "1\otimes(1\otimes copy_Z)"] & X \otimes ((Y \otimes Y) \otimes (Z \otimes Z)) \arrow[d, "1\otimes gr^{-1}_{Y,Z}"] \\
X \otimes (Y \otimes Z) \arrow[rr, "1\otimes copy_{Y\otimes Z}"'] & & X \otimes ((Y \otimes Z) \otimes (Y \otimes Z)),
\end{tikzcd}$$

commutes by (15). The diagram

$$\begin{tikzcd}
X \otimes ((Y \otimes Y) \otimes (Z \otimes Z)) \arrow[r, "copy_X\otimes 1"] \arrow[d, "1\otimes gr^{-1}_{Y,Z}"'] & (X \otimes X) \otimes ((Y \otimes Y) \otimes (Z \otimes Z)) \arrow[d, "1\otimes gr^{-1}_{Y,Z}"] \\
X \otimes ((Y \otimes Z) \otimes (Y \otimes Z)) \arrow[r, "copy_X\otimes 1"'] & (X \otimes X) \otimes ((Y \otimes Z) \otimes (Y \otimes Z)),
\end{tikzcd}$$

commutes by functoriality of $- \otimes -$. The diagram

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "1\otimes copy_{Y,Z}"] \arrow[d, equal] & X \otimes ((Y \otimes Z) \otimes (Y \otimes Z)) \arrow[r, "copy_X\otimes 1"] & (X \otimes X) \otimes ((Y \otimes Z) \otimes (Y \otimes Z)) \arrow[d, "gr_{X,Y\otimes Z}"] \\
X \otimes (Y \otimes Z) \arrow[rr, "copy_{X\otimes(Y\otimes Z)}"'] & & (X \otimes (Y \otimes Z)) \otimes (X \otimes (Y \otimes Z)),
\end{tikzcd}$$

commutes by (15) and functoriality of $- \otimes -$. Putting all the diagrams above together, we have that the following big diagram commutes, writting ";" instead of "⊗" for brevity,

$$\begin{tikzcd}
(X;Y);Z \arrow[rrr, "copy_{(X;Y);Z}"] \arrow[d, equal] & & & ((X;Y);Z);((X;Y);Z) \arrow[d, "gr_{(X;Y);Z}"] \\
(X;Y);Z \arrow[rr, "copy_{X;Y};1"] \arrow[d, equal] & & ((X;Y);(X;Y));Z \arrow[r, "1;copy_Z"] & ((X;Y);(X;Y));(Z;Z) \arrow[d, "gr_{X,Y};1"] \\
(X;Y);Z \arrow[r, "(1;copy_Y);1"] \arrow[d, "a_{X,Y,Z}"', "\cong"] & (X;(Y;Y));Z \arrow[r, "(copy_X;1);1"] \arrow[d, "\cong"', "a_{X,(Y;Y),Z}"] & ((X;X);(Y;Y));Z \arrow[r, "1;copy_Z"] \arrow[d, "\cong"', "a_{(X;X),(Y;Y),Z}"] & ((X;X);(Y;Y));(Z;Z) \arrow[d, "\cong"', "a_{(X;X),(Y;Y),(Z;Z)}"] \\
X;(Y;Z) \arrow[r, "1;(copy_Y;1)"] \arrow[d, equal] & X;((Y;Y);Z) \arrow[r, "copy_X;1"] \arrow[d, equal] & (X;X);((Y;Y);Z) \arrow[r, "1;(1;copy_Z)"] & (X;X);((Y;Y);(Z;Z)) \arrow[d, equal] \\
X;(Y;Z) \arrow[r, "1;(copy_Y;1)"] \arrow[d, equal] & X;((Y;Y);Z) \arrow[r, "1;(1;copy_Z)"] & X;((Y;Y);(Z;Z)) \arrow[r, "copy_X;1"] \arrow[d, "1;gr^{-1}_{Y,Z}"] & (X;X);((Y;Y);(Z;Z)) \arrow[d, "1;gr^{-1}_{Y,Z}"] \\
X;(Y;Z) \arrow[rr, "1;copy_{Y,Z}"] \arrow[d, equal] & & X;((Y;Z);(Y;Z)) \arrow[r, "copy_X;1"] & (X;X);((Y;Z);(Y;Z)) \arrow[d, "gr_{X,Y;Z}"] \\
X;(Y;Z) \arrow[rrr, "copy_{X;(Y;Z)}"'] & & & (X;(Y;Z));(X;(Y;Z)).
\end{tikzcd}$$

The left vertical composition is exactly equal to $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$. And by Theorem 2.3, the right vertical composition morphism is $a_{X,Y,Z} \otimes a_{X,Y,Z} : ((X \otimes Y) \otimes Z) \otimes ((X \otimes Y) \otimes Z) \to (X \otimes (Y \otimes Z)) \otimes (X \otimes (Y \otimes Z))$, i.e.

$$a_{X,Y,Z} \otimes a_{X,Y,Z} = gr_{X,Y \otimes Z} \circ (1_{X \otimes X} \otimes gr^{-1}_{Y,Z}) \circ a_{X \otimes X, Y \otimes Y, Z \otimes Z} \circ (gr_{X,Y} \otimes 1_{Z \otimes Z}) \circ gr_{(X \otimes Y) \otimes Z},$$

so that

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "copy_{(X \otimes Y);Z}"] \arrow[d, "a_{X,Y,Z}"'] \arrow[dr, phantom, "\circlearrowleft"] & ((X \otimes Y) \otimes Z) \otimes ((X \otimes Y) \otimes Z) \arrow[d, "a_{X,Y,Z} \otimes a_{X,Y,Z}"] \\
X \otimes (Y \otimes Z) \arrow[r, "copy_{X \otimes (Y \otimes Z)}"'] & (X \otimes (Y \otimes Z)) \otimes (X \otimes (Y \otimes Z)),
\end{tikzcd}$$

which means that $a_{X,Y,Z}$ is deterministic.

□

Since all Markov categories contains a subcategory of deterministic morphisms, this gives an idea of another way of constructing Markov categories, i.e., defining the determinisms and then expanding the category to non-deterministic morphisms. For example, the subcategory of deterministic morphisms of *FinStoch* is *FinSet*.

3.2 The 2-category of Markov categories

We are now interested in how to define functors between Markov categories, preserving the structure. And then we can construct a category out of it, that we will call *Markov*.

Definition 3.9. [2, 10.14] The collection of Markov categories makes up a 2-category *Markov*, where:

- objects are Markov categories;
- 1-morphisms $F : \mathcal{C} \to \mathcal{D}$ are *Markov functors*, by which we mean strong symmetric monoidal functors which preserves the comonoids in the sense that the diagram

$$\begin{tikzcd}
& FX \arrow[dl, "copy_{FX}"'] \arrow[dr, "F(copy_X)"] & \\
FX \otimes FX \arrow[rr, "\cong", "\nabla_{X,X}"'] & & F(X \otimes X),
\end{tikzcd}$$

commutes for every $X \in \mathcal{C}$, where the horizontal arrow is the relevant structure morphism of F. Functors that satisfy this property are called *comonoid functors*;

- 2-morphisms $t : F \Rightarrow G$ for $F, G : \mathcal{C} \to \mathcal{D}$ are monoidal natural transformations with deterministic components, i.e., for every X in $\mathcal{C}$, $t_X : F(X) \to G(X)$ is deterministic, and for every X_1, X_2 in $\mathcal{C}$,

$$\begin{tikzcd} F(X_1) \otimes F(X_2) \arrow[r, "t_{X_1} \otimes t_{X_2}"] \arrow[d, "\nabla_{F_{X_1, X_2}}"'] & G(X_1) \otimes G(X_2) \arrow[d, "\nabla_{G_{X_1, X_2}}"] \\ F(X_1 \otimes X_2) \arrow[r, "t_{X_1 \otimes X_2}"'] & G(X_1 \otimes X_2), \end{tikzcd}$$

commutes.

- A comonoid functor $F : \mathcal{C} \to \mathcal{D}$ is a *comonoid equivalence* if there exists a comonoid functor $G : \mathcal{D} \to \mathcal{C}$ and isomorphic 2-morphisms $i : Id_{\mathcal{D}} \cong GF$ and $j : Id_{\mathcal{D}} \cong FG$.

There is a necessary and sufficient condition for a functor to be a comonoid equivalence, which simplify their construction and the verifications.

Proposition 3.10. *[2, 10.16]. A Markov functor $F : \mathcal{C} \to \mathcal{D}$ is a comonoid equivalence if and only if it is an equivalence of categories, and for every object $Y \in \mathcal{D}$ there is $X \in \mathcal{C}$ and a morphism $i : Y \xrightarrow{\cong} FX$ which is deterministic, meaning that the diagram*

$$\begin{tikzcd} Y \arrow[r, "i", "\cong"'] \arrow[d, "copy_Y"'] & FX \arrow[d, "copy_{FX}"] \\ Y \otimes Y \arrow[r, "\cong", "i \otimes i"'] & FX \otimes FX, \end{tikzcd}$$

commutes.

Proof. Suppose that F is a comonoid equivalence. There exists a comonoid functor $G : \mathcal{D} \to \mathcal{C}$ and a 2-isomorphism in *Markov* $i : Id_{\mathcal{D}} \cong GF$ so that for every Y in $\mathcal{D}$, we can take $X := GY$ and since i has deterministic components, the following diagram commutes,

$$\begin{tikzcd} Y \arrow[r, "i_Y", "\cong"'] \arrow[d, "copy_Y"'] & FGY \arrow[d, "copy_{FGY}"] \\ Y \otimes Y \arrow[r, "\cong", "i_Y \otimes i_Y"'] & FGY \otimes FGY. \end{tikzcd}$$

Conversely, suppose that F is an equivalence of categories and that for every object Y in $\mathcal{D}$, there is X in $\mathcal{C}$ and an isomorphism

$$i_Y : Y \xrightarrow{\cong} FX, \tag{25}$$

which is deterministic. There exists $H : \mathcal{D} \to \mathcal{C}$ being right adjoint to F via the natural isomorphisms

$$\eta : 1_{\mathcal{C}} \cong HF,$$
$$\epsilon : FH \cong 1_{\mathcal{D}}.$$

We can construct a functor $G : \mathcal{D} \to \mathcal{C}$, defined as follows.

- On objects, for all Y in $\mathcal{D}$, let $G(Y) := X$, with X the object in (25).
- On morphisms, for all $f : Y_1 \to Y_2$ in $\mathcal{D}$, let $G(f) : X_1 \to X_2$ be defined by the composition

$$X_1 \xrightarrow{\eta_{X_1}} HF(X_1) \xrightarrow{H(i_{Y_1}^{-1})} H(Y_1) \xrightarrow{H(f)} H(Y_2) \xrightarrow{H(i_{Y_2})} HF(X_2) \xrightarrow{\eta_{X_2}^{-1}} X_2,$$

where $i_Y^{-1} : FX \to Y$ is the inverse of i_Y. Then G is a functor since for all Y in $\mathcal{D}$, let $X := G(Y)$ the corresponding object in $\mathcal{C}$,

$$\begin{aligned} G(1_Y) &= \eta_X^{-1} \circ H(i_Y) \circ H(1_Y) \circ H(i_Y^{-1}) \circ \eta_X \\ &= \eta_X^{-1} \circ H(i_Y \circ 1_Y \circ i_Y^{-1}) \circ \eta_X \\ &= \eta_X^{-1} \circ 1_{H(FX)} \circ \eta_X \\ &= 1_X = 1_{G(Y)}, \end{aligned}$$

by functoriality of H, and for all morphisms $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_3$ in $\mathcal{D}$, let $X_1 := G(Y_1)$, $X_2 := G(Y_2)$, $X_3 := G(Y_3)$ the corresponding objects in C,

$$\begin{aligned} G(g) \circ G(f) =& \eta_{X_3}^{-1} \circ H(i_{Y_3}) \circ H(g) \circ H(i_{Y_2}^{-1}) \circ \eta_{X_2} \circ \eta_{X_2}^{-1} \circ H(i_{Y_2}) \circ H(f) \circ H(i_{Y_1}^{-1}) \circ \eta_{X_1} \\ =& \eta_{X_3}^{-1} \circ H(i_{Y_3}) \circ H(g \circ f) \circ H(i_{Y_1}^{-1}) \circ \eta_{X_1}, \end{aligned}$$

again by functoriality of H.

We now shox that F and G form an equivalence. We show that the morphisms i_Y define a natural isomorphism $i : 1_{\mathcal{D}} \Rightarrow FG$, i.e., for every morphism $f : Y_1 \to Y_2$ in $\mathcal{D}$, with $X_1 := G(Y_1)$ and $X_2 := G(Y_2)$, we want to show that the following diagram commutes,

$$\begin{tikzcd} Y_1 \arrow[r, "i_{Y_1}"] \arrow[d, "f"'] & F(X_1) = FG(Y_1) \arrow[r, "F(\eta_{X_1})"] \arrow[d, "FG(f)"] & FHF(X_1) \arrow[r, "FH(i_{Y_1}^{-1})"] \arrow[d, phantom, "\circlearrowleft"] & FH(Y_1) \arrow[d, "FH(f)"] \\ Y_2 \arrow[r, "i_{Y_2}"] & F(X_2) = FG(Y_2) & FHF(X_2) \arrow[l, "F(\eta_{X_2}^{-1})"] & FH(Y_2). \arrow[l, "FH(i_{Y_2})"] \end{tikzcd}$$

By naturality of η and functoriality of F, the following diagram commutes,

$$\begin{tikzcd} F(X_1) \arrow[r, "F(\eta_{X_1})"] \arrow[d, "i_{Y_1}^{-1} \circ f \circ i_{Y_2}"'] & FHF(X_1) \arrow[d, "FH(i_{Y_1}^{-1} \circ f \circ i_{Y_2})"] \\ F(X_2) & FHF(X_2). \arrow[l, "F(\eta_{X_2}^{-1})"] \end{tikzcd}$$

By functoriality of F and H, the following diagram commutes,

$$\begin{tikzcd} FHF(X_1) \arrow[r, "FH(i_{Y_1}^{-1})"] \arrow[d, "FH(i_{Y_1}^{-1} \circ f \circ i_{Y_2})"'] & FH(Y_1) \arrow[d, "FH(f)"] \\ FHF(X_2) & FH(Y_2). \arrow[l, "FH(i_{Y_2})"] \end{tikzcd}$$

Putting the diagrams above together, we have that the following big diagram commutes,

$$\begin{tikzcd} Y_1 \arrow[r, "i_{Y_1}"] \arrow[d, "f"'] & F(X_1) \arrow[r, "F(\eta_{X_1})"] \arrow[d, "i_{Y_1}^{-1} \circ f \circ i_{Y_2}"] & FHF(X_1) \arrow[r, "FH(i_{Y_1}^{-1})"] \arrow[d, "FH(i_{Y_1}^{-1} \circ f \circ i_{Y_2})"] & FH(Y_1) \arrow[d, "FH(f)"] \\ Y_2 \arrow[r, "i_{Y_2}"'] & F(X_2) & FHF(X_2) \arrow[l, "F(\eta_{X_2}^{-1})"] & FH(Y_2), \arrow[l, "FH(i_{Y_2})"] \end{tikzcd}$$

establishing naturality of i.

Now we need to define a natural isomorphim $j : 1_C \Rightarrow GF$ inducing with $i^{-1} : FG \Rightarrow 1_{\mathcal{D}}$ an adjunction on the pair $F : C \to \mathcal{D}$, $G : \mathcal{D} \to C$. Define $j : 1_C \Rightarrow GF$ on X in C by the composition

$$X \xrightarrow{\eta_X} HFX \xrightarrow{H(i_{FX})} HFGFX \xrightarrow{\eta_{GFX}^{-1}} GFX$$

this defines a natural isomorphism, since all the morphisms involved are natural ismorphisms. The triangular equalities are satisfied. By naturality of ϵ and the triangle equalities of the pair η, ϵ, the following diagram commutes,

$$\begin{tikzcd} FX \arrow[r, "F(\eta_X)"] \arrow[rrr, bend left, "F(j_X)"] \arrow[dr, equal] & FHFX \arrow[r, "FH(i_{FX})"] \arrow[d, "\epsilon_{FX}"] & FHFGFX \arrow[r, "F(\eta_{GFX}^{-1})"] \arrow[d, "\epsilon_{FGFX}"] & FGFX \arrow[d, "i_{FX}^{-1}"] \\ & FX \arrow[r, "i_{FX}"'] & FGFX \arrow[r, "i_{FX}^{-1}"'] \arrow[ur, equal] & FX, \end{tikzcd}$$

establishing the first triangle equality. By definition of G, $G(i_Y)$ is the composition

$$\begin{tikzcd} GY \arrow[r, "\eta_{GY}"] & HFGY \arrow[r, "H(i_Y^{-1})"] \arrow[rr, bend right, equal] & GY \arrow[r, "H(i_Y)"] & HFGY \arrow[r, "H(i_{FGY})"] & HFGFGY \arrow[r, "\eta^{-1}_{GFGY}"] & GFGY, \end{tikzcd}$$

and by definition of j, j_{GY} is the composition

$$\begin{tikzcd} GY \arrow[r, "\eta_{GY}"] & HFGY \arrow[r, "H(i_{FGY})"] & HFGFGY \arrow[r, "\eta^{-1}_{GFGY}"] & GFGY, \end{tikzcd}$$

so that $j_{GY} = G(i_Y)$ and the following diagram commutes,

$$\begin{tikzcd} GY \arrow[r, "j_{GY}"] \arrow[dr, equal] & GFGY \arrow[d, "G(i_Y^{-1})"] \\ & GY, \end{tikzcd}$$

establishing the second triangle equality.

We now show that $i : 1_{\mathcal{D}} \Rightarrow FG$ and $j : 1_{C} \Rightarrow GF$ are 2-isomorphisms in the category *Markov*. By assumption, i has deterministic components. We need to prove that it is a monoidal natural transformation. We did not define the isomorphism $\nabla_{G_{Y_1,Y_2}} : G(Y_1) \otimes G(Y_2) \to G(Y_1 \otimes Y_2)$, so we will define it and show that it makes the following diagram commute,

$$\begin{tikzcd} Y_1 \otimes Y_2 \arrow[r, "i_{Y_1} \otimes i_{Y_2}"] \arrow[dd, equal] & FG(Y_1) \otimes FG(Y_2) \arrow[d, "\nabla_{F_{G(Y_1),G(Y_2)}}"] \\ & F(G(Y_1) \otimes G(Y_2)) \arrow[d, "F\nabla_{G_{Y_1,Y_2}}"] \\ Y_1 \otimes Y_2 \arrow[r, "i_{Y_1 \otimes Y_2}"'] & FG(Y_1 \otimes Y_2). \end{tikzcd}$$

For all couples of objects Y_1, Y_2 in $\mathcal{D}$, $\nabla_{G_{Y_1,Y_2}} : G(Y_1) \otimes G(Y_2) \to G(Y_1 \otimes Y_2)$ is defined by the composition

$$\begin{tikzcd} G(Y_1) \otimes G(Y_2) \arrow[r, "j_{G(Y_1) \otimes G(Y_2)}"] \arrow[d, "\nabla_{G_{Y_1,Y_2}}"'] & GF(G(Y_1) \otimes G(Y_2)) \arrow[r, "G(\nabla^{-1}_{F_{G(Y_1),G(Y_2)}})"] & G(FG(Y_1) \otimes FG(Y_2)) \arrow[d, "G(i_{Y_1}^{-1} \otimes i_{Y_2}^{-1})"] \\ G(Y_1 \otimes Y_2) & GFG(Y_1 \otimes Y_2) \arrow[l, "j^{-1}_{G(Y_1 \otimes Y_2)}"'] & G(Y_1 \otimes Y_2). \arrow[l, "G(i_{Y_1 \otimes Y_2})"'] \arrow[ll, bend left, equal] \end{tikzcd}$$

This defines a natural isomorphism by composition of natural isomorphisms. Moreover, the following diagram commutes, using twice the triangle equalities and three times naturality of i:

$$\begin{tikzcd} Y_1 \otimes Y_2 \arrow[rr, "i_{Y_1} \otimes i_{Y_2}"] \arrow[d, "i_{Y_1 \otimes Y_2}"'] \arrow[ddrr, "i_{Y_1 \otimes Y_2}"] & & FG(Y_1) \otimes FG(Y_2) \arrow[rr, "\nabla_{F_{G(Y_1),G(Y_2)}}"] \arrow[ddrr, "i_{FGY_1 \otimes FGY_2}"'] & & F(G(Y_1) \otimes G(Y_2)) \arrow[d, "i_{F(GY_1 \otimes GY_2)} = F(j_{G(Y_1) \otimes G(Y_2)})"] \\ FG(Y_1 \otimes Y_2) & & \circlearrowleft & \circlearrowleft & FGF(G(Y_1) \otimes G(Y_2)) \arrow[d, "FG(\nabla^{-1}_{F_{G(Y_1),G(Y_2)}})"] \\ FGFG(Y_1 \otimes Y_2) \arrow[u, "F(j^{-1}_{G(Y_1 \otimes Y_2)}) = i^{-1}_{FG(Y_1 \otimes Y_2)}"] \arrow[r, phantom, "\circlearrowleft"] & & FG(Y_1 \otimes Y_2) \arrow[ll, "FG(i_{Y_1 \otimes Y_2})"] & & FG(FG(Y_1) \otimes FG(Y_2)), \arrow[ll, "FG(i_{Y_1}^{-1} \otimes i_{Y_2}^{-1})"] \end{tikzcd}$$

making the following diagram commute, by definition of ∇_G,

$$\begin{tikzcd} Y_1 \otimes Y_2 \arrow[r, "i_{Y_1} \otimes i_{Y_2}"] \arrow[dd, equal] & FG(Y_1) \otimes FG(Y_2) \arrow[d, "\nabla_{F_{G(Y_1),G(Y_2)}}"] \\ & F(G(Y_1) \otimes G(Y_2)) \arrow[r, "F(j_{G(Y_1) \otimes G(Y_2)})"] \arrow[d, "F\nabla_{G_{Y_1,Y_2}}"] & FGF(G(Y_1) \otimes G(Y_2)) \arrow[r, "FG(\nabla^{-1}_{F_{G(Y_1),G(Y_2)}})"] & FG(FG(Y_1) \otimes FG(Y_2)) \arrow[d, "FG(i_{Y_1}^{-1} \otimes i_{Y_2}^{-1})"] \\ Y_1 \otimes Y_2 \arrow[r, "i_{Y_1 \otimes Y_2}"'] & FG(Y_1 \otimes Y_2) & FGFG(Y_1 \otimes Y_2) \arrow[l, "F(j^{-1}_{G(Y_1 \otimes Y_2)})"] & FG(Y_1 \otimes Y_2), \arrow[l, "FG(i_{Y_1 \otimes Y_2})"] \end{tikzcd}$$

establishing monoidality of i.

We need to prove that j is monoidal and has deterministic components. For every couple of objects X_1, X_2 in C, the following diagram commutes, by naturality of j for the upper diagram, by naturality of ∇_F combined with functoriality of G for the middle diagram and by triangle equality for the lower one:

$$\begin{tikzcd}
X_1 \otimes X_2 \arrow[r, "j_{X_1} \otimes j_{X_2}"] \arrow[d, "j_{X_1 \otimes X_2}"'] & GF(X_1) \otimes GF(X_2) \arrow[d, "j_{GFX_1 \otimes GFX_2}"] \\
GF(X_1 \otimes X_2) \arrow[r, "GF(j_{X_1} \otimes j_{X_2})"] & GF(GFX_1 \otimes GFX_2) \\
G(FX_1 \otimes FX_2) \arrow[u, "G\nabla_{F_{X_1,X_2}}"] \arrow[r, "G(Fj_{X_1} \otimes Fj_{X_2})"] \arrow[r, bend right, "G(i_{FX_1} \otimes i_{FX_2})"'] & G(FGFX_1 \otimes FGFX_2), \arrow[u, "G\nabla_{F_{GFX_1,GFX_2}}"']
\end{tikzcd}$$

and this gives the monoidality of $j : 1_C \Rightarrow GF$, i.e., that the following diagram commutes, by definition of ∇_G,

$$\begin{tikzcd}
X_1 \otimes X_2 \arrow[r, "j_{X_1} \otimes j_{X_2}"] \arrow[dd, equal] & GF(X_1) \otimes GF(X_2) \arrow[r, "j_{GFX_1, GFX_2}"] \arrow[d, "\nabla_{G_{F(X_1),F(X_2)}}"] & GF(GFX_1 \otimes GFX_2) \arrow[d, "G(\nabla^{-1}_{F_{GFX_1,GFX_2}})"] \\
& G(F(X_1) \otimes F(X_2)) \arrow[d, "G\nabla_{F_{X_1,X_2}}"] & G(FGFX_1 \otimes FGFX_2) \arrow[l, "G(i^{-1}_{FX_1} \otimes i^{-1}_{FX_2})"] \\
X_1 \otimes X_2 \arrow[r, "j_{X_1 \otimes X_2}"'] & GF(X_1 \otimes X_2). &
\end{tikzcd}$$

The components of j are deterministic. For all X in C, the following diagram commutes,

$$\begin{tikzcd}
FX \arrow[rrr, "F(j_X)", "i_{FX}"'] \arrow[dr, "copy_{FX}"] \arrow[ddd, "F(copy_X)"'] & & & FGFX \arrow[dl, "copy_{FGFX}"'] \arrow[ddd, "F(copy_{GFX})"] \\
& FX \otimes FX \arrow[r, "i_{FX} \otimes i_{FX}", "F(j_X) \otimes F(j_X)"'] \arrow[ddl, "\nabla_{F_{X,X}}"] & FGFX \otimes FGFX \arrow[ddr, "\nabla_{F_{GFX \otimes GFX}}"] & \\
& & & \\
F(X \otimes X) \arrow[rrr, "F(j_X \otimes j_X)"'] & & & F(GFX \otimes GFX),
\end{tikzcd}$$

by naturality of ∇_F, by triangle equalities for i and j, and since i has deterministic components and F preserves comonoids. By fully faithfulness of F, the following diagram commutes,

$$\begin{tikzcd}
X \arrow[r, "j_X"] \arrow[d, "copy_X"'] & GFX \arrow[d, "copy_{GFX}"] \\
X \otimes X \arrow[r, "j_X \otimes j_X"'] & GFX \otimes GFX,
\end{tikzcd}$$

this yields that j_X is deterministic. We have proven that i and j are 2-isomorphisms in *Markov*.

We now show that $G : \mathcal{D} \to C$ defined above is a 1-morphism in *Markov*. We can define a unit on G so that it becomes a strong symmetric monoidal functor. If $\xi_F : I_{\mathcal{D}} \to F(I_C)$ is the unit for F, we can define the unit for G by its image on F since F is fully faithful,

$$F(\xi_G) : F(I_C) \xrightarrow{\xi_F^{-1}} I_{\mathcal{D}} \xrightarrow{i_{I_{\mathcal{D}}}} FG(I_{\mathcal{D}}). \tag{26}$$

Let us prove associativity of G. For every triple of objects Y_1, Y_2, Y_3 in $\mathcal{D}$, the following diagrams commute by naturality of ∇_F

$$\begin{tikzcd}
 & F((GY_1 \otimes GY_2) \otimes GY_3) \arrow[ddl, bend right, "F(\nabla_{G_{Y_1,Y_2}} \otimes 1_{GY_3})"'] \\
 & F(GY_1 \otimes GY_2) \otimes FGY_3 \arrow[u, "\nabla_{F_{GY_1 \otimes GY_2, GY_3}}"'] \arrow[dd, "F\nabla_{G_{Y_1,Y_2}} \otimes F1_{GY_3}"] \arrow[l, phantom, "\circlearrowleft"] \\
F(G(Y_1 \otimes Y_2) \otimes GY_3) & \\
 & FG(Y_1 \otimes Y_2) \otimes FGY_3 \arrow[ul, "\nabla_{F_{G(Y_1 \otimes Y_2), GY_3}}"]
\end{tikzcd},$$

$$\begin{tikzcd}
F(GY_1 \otimes (GY_2 \otimes GY_3)) \arrow[ddr, bend left, "F(1_{GY_1} \otimes \nabla_{G_{Y_2,Y_3}})"] & \\
FGY_1 \otimes F(GX_2 \otimes GY_3) \arrow[u, "\nabla_{F_{GY_1, GY_2 \otimes GY_3}}"] \arrow[dd, "F1_{GY_1} \otimes F\nabla_{G_{Y_2,Y_3}}"'] \arrow[r, phantom, "\circlearrowleft"] & {} \\
 & F(G(Y_1) \otimes G(Y_2 \otimes Y_3)) \\
FGY_1 \otimes FG(Y_2 \otimes Y_3) \arrow[ur, "\nabla_{F_{GY_1, G(Y_2 \otimes Y_3)}}"'] &
\end{tikzcd}.$$

Since F is monoidal, the following diagram commutes,

$$\begin{tikzcd}
F((GY_1 \otimes GY_2) \otimes GY_3) \arrow[r, "F(\alpha_{\mathcal{C}_{GY_1, GY_2, GY_3}})"] & F(GY_1 \otimes (GY_2 \otimes GY_3)) \\
F(GY_1 \otimes GY_2) \otimes FGY_3 \arrow[u, "\nabla_{F_{GY_1 \otimes GY_2, GY_3}}"] & FGY_1 \otimes F(GX_2 \otimes GY_3) \arrow[u, "\nabla_{F_{GY_1, GY_2; GY_3}}"] \\
(FGY_1 \otimes FGY_2) \otimes FGY_3 \arrow[u, bend left, "\nabla_{F_{GY_1,GY_2}} \otimes 1_{FGY_3}"'] \arrow[r, "\alpha_{\mathcal{D}_{FGY_1, FGY_2, FGY_3}}"'] \arrow[ur, phantom, "\circlearrowleft"] & FGY_1 \otimes (FGY_2 \otimes FGY_3). \arrow[u, bend right, "\nabla_{F_{GY_1,GY_2}} \otimes 1_{FGY_3}"]
\end{tikzcd}$$

By monoidality of i, the following diagrams commute,

$$
\begin{tikzcd}
F(GY_1 \otimes GY_2) \otimes FGY_3 \arrow[d, "F\nabla_{G_{Y_1,Y_2}} \otimes F1_{GY_3}"'] \\
FG(Y_1 \otimes Y_2) \otimes FGY_3 \\
(FGY_1 \otimes FGY_2) \otimes FGY_3 \arrow[uu, bend right=60, "\nabla_{F_{GY_1,GY_2}} \otimes 1_{FGY_3}"'] \\
(Y_1 \otimes Y_2) \otimes Y_3, \arrow[u, "(i_{Y_1} \otimes i_{Y_2}) \otimes i_{Y_3}"] \arrow[uu, bend left=60, "i_{Y_1 \otimes Y_2} \otimes i_{Y_3}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
FGY_1 \otimes F(GY_2 \otimes GY_3) \arrow[d, "F1_{GY_1} \otimes F\nabla_{G_{Y_2,Y_3}}"] \\
FGY_1 \otimes FG(Y_2 \otimes Y_3) \\
FGY_1 \otimes (FGY_2 \otimes FGY_3) \arrow[uu, bend left=60, "1_{FGY_1} \otimes \nabla_{F_{GY_2,GY_3}}"] \\
Y_1 \otimes (Y_2 \otimes Y_3), \arrow[u, "i_{Y_1} \otimes (i_{Y_2} \otimes i_{Y_3})"'] \arrow[uu, bend right=60, "i_{Y_1} \otimes i_{Y_2 \otimes Y_3}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(G(Y_1 \otimes Y_2) \otimes GY_3) \arrow[ddd, "F\nabla_{G_{Y_1 \otimes Y_2, Y_3}}"'] & \\
& FG(Y_1 \otimes Y_2) \otimes FGY_3 \arrow[ul, "\nabla_{F_{G(Y_1 \otimes Y_2), GY_3}}"] \\
& (Y_1 \otimes Y_2) \otimes Y_3 \arrow[u, bend left=60, "i_{Y_1 \otimes Y_2} \otimes i_{Y_3}"] \arrow[dl, "i_{(Y_1 \otimes Y_2) \otimes Y_3}"] \\
FG((Y_1 \otimes Y_2) \otimes Y_3), &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& F(G(Y_1) \otimes G(Y_2 \otimes Y_3)) \arrow[ddd, "F\nabla_{G_{Y_1, Y_2 \otimes Y_3}}"] \\
FGY_1 \otimes FG(Y_2 \otimes Y_3) \arrow[ur, "\nabla_{F_{GY_1, G(Y_2 \otimes Y_3)}}"'] & \\
Y_1 \otimes (Y_2 \otimes Y_3) \arrow[u, bend right=60, "i_{Y_1} \otimes i_{Y_2 \otimes Y_3}"'] \arrow[dr, "i_{Y_1 \otimes (Y_2 \otimes Y_3)}"'] & \\
& FG(Y_1 \otimes (Y_2 \otimes Y_3)).
\end{tikzcd}
$$

By naturality of α, the following diagram commutes,

$$
\begin{tikzcd}
(FGY_1 \otimes FGY_2) \otimes FGY_3 \arrow[r, "\alpha_{\mathcal{D}}"] & FGY_1 \otimes (FGY_2 \otimes FGY_3) \\
(Y_1 \otimes Y_2) \otimes Y_3 \arrow[u, "(i_{Y_1} \otimes i_{Y_2}) \otimes i_{Y_3}"] \arrow[r, "\alpha_{\mathcal{D}_{Y_1,Y_2,Y_3}}"] & Y_1 \otimes (Y_2 \otimes Y_3). \arrow[u, "i_{Y_1} \otimes (i_{Y_2} \otimes i_{Y_3})"']
\end{tikzcd}
$$

By naturality of i, the following diagram commutes,

$$
\begin{tikzcd}
& (Y_1 \otimes Y_2) \otimes Y_3 \arrow[r, "\alpha_{\mathcal{D}_{Y_1,Y_2,Y_3}}"] \arrow[dl, "i_{(X\otimes Y)\otimes Z}"'] & Y_1 \otimes (Y_2 \otimes Y_3) \arrow[dr, "i_{X\otimes(Y\otimes Z)}"] & \\
FG((Y_1 \otimes Y_2) \otimes Y_3) \arrow[rrr, "FG(\alpha_{\mathcal{D}_{Y_1,Y_2,Y_3}})"'] & & & FG(Y_1 \otimes (Y_2 \otimes Y_3)).
\end{tikzcd}
$$

Putting these diagrams together, we have that the following big diagram commutes

$$
\begin{tikzcd}
& F((GY_1;GY_2);GY_3) \arrow[r, "F(\alpha_{\mathcal{C}_{GY_1,GY_2,GY_3}})"] \arrow[dl, bend right, "F(\nabla_{G_{Y_1,Y_2}};1_{GY_3})"'] & F(GY_1;(GY_2;GY_3)) \arrow[dr, bend left, "F(1_{GY_1};\nabla_{G_{Y_2,Y_3}})"] & \\
& F(GY_1;GY_2);FGY_3 \arrow[u, "\nabla_{F_{GY_1;GY_2,GY_3}}"] \arrow[dd, "F\nabla_{G_{Y_1,Y_2}};F1_{GY_3}"'] & FGY_1;F(GY_2;GY_3) \arrow[u, "\nabla_{F_{GY_1,GY_2;GY_3}}"'] \arrow[dd, "F1_{GY_1};F\nabla_{G_{Y_2,Y_3}}"] & \\
F(G(Y_1;Y_2);GY_3) \arrow[dddd, "F\nabla_{G_{Y_1;Y_2,Y_3}}"'] & & & F(GY_1;G(Y_2;Y_3)) \arrow[dddd, "F\nabla_{G_{Y_1,Y_2;Y_3}}"] \\
& FG(Y_1;Y_2);FGY_3 \arrow[ul, "\nabla_{F_{G(Y_1;Y_2),GY_3}}"] & FGY_1;FG(Y_2;Y_3) \arrow[ur, "\nabla_{F_{GY_1,G(Y_2;Y_3)}}"'] & \\
& (FGY_1;FGY_2);FGY_3 \arrow[r, "\alpha_{\mathcal{D}}"] \arrow[uuu, bend right, "\nabla_F;1"'] & FGY_1;(FGY_2;FGY_3) \arrow[uuu, bend left, "1;\nabla_F"] & \\
& (Y_1;Y_2);Y_3 \arrow[u, "(i_{Y_1};i_{Y_2});i_{Y_3}"] \arrow[r, "\alpha_{\mathcal{D}_{Y_1,Y_2,Y_3}}"] \arrow[uu, bend left=60, "i_{Y_1;Y_2};i_{Y_3}"] \arrow[dl, "i_{(Y_1;Y_2);Y_3}"] & Y_1;(Y_2;Y_3) \arrow[u, "i_{Y_1};(i_{Y_2};i_{Y_3})"'] \arrow[uu, bend right=60, "i_{Y_1};i_{Y_2;Y_3}"'] \arrow[dr, "i_{Y_1;(Y_2;Y_3)}"'] & \\
FG((Y_1;Y_2);Y_3) \arrow[rrr, "FG(\alpha_{\mathcal{D}_{Y_1,Y_2,Y_3}})"'] & & & FG(Y_1;(Y_2;Y_3)),
\end{tikzcd}
$$

where we write ";" instead of $\otimes$ for simplicity, which yields associativity of G since F is faithful.

Let us now prove the unitality of G. Again, it is enough to look at the image under F by faithfulness of F. For every Y in $\mathcal{D}$, we want the following diagrams to commute:

$$
\begin{tikzcd}
F(I_{\mathcal{C}} \otimes GY) \arrow[r, "F(\xi_G \otimes 1_{GY})"] \arrow[d, "F(\lambda_{\mathcal{C}_{GY}})"'] & F(GI_{\mathcal{D}} \otimes GY) \arrow[d, "F\nabla_{G_{I_{\mathcal{D}},Y}}"] \\
FGY & FG(I_{\mathcal{D}} \otimes Y), \arrow[l, "FG(\lambda_{\mathcal{D}_Y})"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(GY \otimes I_{\mathcal{C}}) \arrow[r, "F(1_{GY} \otimes \xi_G)"] \arrow[d, "F(\rho_{\mathcal{C}_{GY}})"'] & F(GY \otimes GI_{\mathcal{D}}) \arrow[d, "F\nabla_{G_{Y,I_{\mathcal{D}}}}"] \\
FGY & FG(Y \otimes I_{\mathcal{D}}). \arrow[l, "FG(\rho_{\mathcal{D}_Y})"]
\end{tikzcd}
$$

Since ∇_F is natural, we have that the following diagrams commute,

$$
\begin{tikzcd}
F(I_{\mathcal{C}} \otimes GY) \arrow[rrr, "F(\xi_G \otimes 1_{GY})"] & & & F(GI_{\mathcal{D}} \otimes GY) \\
& F(I_{\mathcal{C}}) \otimes FGY \arrow[ul, "\nabla_{F_{I_{\mathcal{C}},GY}}"] \arrow[r, "F(\xi_G)\otimes 1_{FGY}"] & FGI_{\mathcal{D}} \otimes FGY, \arrow[ur, "\nabla_{F_{GI_{\mathcal{D}},GY}}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(GY \otimes I_{\mathcal{C}}) \arrow[rrr, "F(1_{GY} \otimes \xi_G)"] & & & F(GY \otimes GI_{\mathcal{D}}) \\
& FGY \otimes F(I_{\mathcal{C}}) \arrow[ul, "\nabla_{F_{GY,I_{\mathcal{C}}}}"] \arrow[r, "1_{FGY}\otimes F(\xi_G)"] & FGY \otimes FGI_{\mathcal{D}}. \arrow[ur, "\nabla_{F_{GY,GI_{\mathcal{D}}}}"'] &
\end{tikzcd}
$$

Since F is monoidal, the following diagrams commute,

$$
\begin{tikzcd}
F(I_{\mathcal{C}} \otimes GY) \arrow[ddd, "F(\lambda_{\mathcal{C}_{GY}})"'] & & \\
& & F(I_{\mathcal{C}}) \otimes FGY, \arrow[ull, "\nabla_{F_{I_{\mathcal{C}},GY}}"'] \\
& & I_{\mathcal{D}}; FGY \arrow[u, "\xi_F \otimes 1_{FGY}"] \arrow[dll, "\lambda_{\mathcal{D}_{FGY}}"'] \\
FGY & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(GY \otimes I_{\mathcal{C}}) \arrow[ddd, "F(\rho_{\mathcal{C}_{GY}})"'] & & \\
& & FGY \otimes F(I_{\mathcal{C}}) \arrow[ull, "\nabla_{F_{GY,I_{\mathcal{C}}}}"'] \\
& & FGY \otimes I_{\mathcal{D}}. \arrow[u, "1_{FGY} \otimes \xi_F"] \arrow[dll, "\rho_{\mathcal{D}_{FGY}}"'] \\
FGY & &
\end{tikzcd}
$$

By defintion of ξ_G in (26), the following diagrams commute,

$$
\begin{tikzcd}
F(I_{\mathcal{C}}) \otimes FGY \arrow[r, "F(\xi_G) \otimes 1_{FGY}"] & FGI_{\mathcal{D}} \otimes FGY \\
I_{\mathcal{D}} \otimes FGY, \arrow[u, "\xi_F \otimes 1_{FGY}"] \arrow[ur, "i_{I_{\mathcal{D}}} \otimes 1_{FGY}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
FGY \otimes F(I_{\mathcal{C}}) \arrow[r, "1_{FGY} \otimes F(\xi_G)"] & FGY \otimes FGI_{\mathcal{D}} \\
FGY \otimes I_{\mathcal{D}}. \arrow[u, "1_{FGY} \otimes \xi_F"] \arrow[ur, "1_{FGY} \otimes i_{I_{\mathcal{D}}}"'] &
\end{tikzcd}
$$

By naturality of λ, the following diagrams commute,

$$
\begin{tikzcd}
& I_{\mathcal{D}} \otimes FGY \arrow[ddl, "\lambda_{\mathcal{D}_{FGY}}"'] & \\
& Y \arrow[dl, "i_Y"] & I_{\mathcal{D}} \otimes Y, \arrow[l, "\lambda_{\mathcal{D}_Y}"] \arrow[ul, "1_{I_{\mathcal{D}}} \otimes i_Y"'] \\
FGY & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& FGY \otimes I_{\mathcal{D}} \arrow[ddl, "\rho_{\mathcal{D}_{FGY}}"'] & \\
& Y \arrow[dl, "i_Y"] & Y \otimes I_{\mathcal{D}}. \arrow[l, "\rho_{\mathcal{D}_Y}"] \arrow[ul, "i_Y \otimes 1_{I_{\mathcal{D}}}"'] \\
FGY & &
\end{tikzcd}
$$

By monoidality of i, the following diagrams commute,

$$
\begin{tikzcd}
& F(GI_{\mathcal{D}} \otimes GY) \arrow[ddd, "F\nabla_{G_{I_{\mathcal{D}},Y}}"] \\
FGI_{\mathcal{D}} \otimes FGY \arrow[ur, "\nabla_{F_{GI_{\mathcal{D}},GY}}"] & \\
I_{\mathcal{D}} \otimes Y \arrow[u, "i_{I_{\mathcal{D}}} \otimes i_Y"] \arrow[dr, "i_{I_{\mathcal{D}} \otimes Y}"'] & \\
& FG(I_{\mathcal{D}} \otimes Y),
\end{tikzcd}
\qquad
\begin{tikzcd}
& F(GY \otimes GI_{\mathcal{D}}) \arrow[ddd, "F\nabla_{G_{Y,I_{\mathcal{D}}}}"] \\
FGY \otimes FGI_{\mathcal{D}} \arrow[ur, "\nabla_{F_{GY,GI_{\mathcal{D}}}}"] & \\
Y \otimes I_{\mathcal{D}} \arrow[u, "i_{i_Y \otimes I_{\mathcal{D}}}"] \arrow[dr, "i_{Y \otimes I_{\mathcal{D}}}"'] & \\
& FG(Y \otimes I_{\mathcal{D}}).
\end{tikzcd}
$$

By naturality of i, the following diagrams commute,

$$
\begin{tikzcd}
& Y \arrow[dl, "i_Y"'] & I_{\mathcal{D}} \otimes Y \arrow[l, "\lambda_{\mathcal{D}_Y}"] \arrow[dr, "i_{I_{\mathcal{D}} \otimes Y}"] & \\
FGY & & & FG(I_{\mathcal{D}} \otimes Y), \arrow[lll, "FG(\lambda_{\mathcal{D}_Y})"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
& Y \arrow[dl, "i_Y"'] & Y \otimes I_{\mathcal{D}} \arrow[l, "\rho_{\mathcal{D}_Y}"] \arrow[dr, "i_{Y \otimes I_{\mathcal{D}}}"] & \\
FGY & & & FG(Y \otimes I_{\mathcal{D}}). \arrow[lll, "FG(\rho_{\mathcal{D}_Y})"]
\end{tikzcd}
$$

Putting these diagrams all together, we have that the following two big diagrams commute,

$$
\begin{tikzcd}
F(I_{\mathcal{C}}; GY) \arrow[rrr, "F(\xi_G;1_{GY})"] \arrow[dddd, "F(\lambda_{\mathcal{C}_{GY}})"'] & & & F(GI_{\mathcal{D}}; GY) \arrow[dddd, "F\nabla_{G_{I_{\mathcal{D}}},Y}"] \\
& F(I_{\mathcal{C}}); FGY \arrow[ul, "\nabla_{F_{I_{\mathcal{C}}},GY}"'] \arrow[r, "F(\xi_G);1_{FGY}"] & FGI_{\mathcal{D}}; FGY \arrow[ur, "\nabla_{F_{GI_{\mathcal{D}}},GY}"] & \\
& I_{\mathcal{D}}; FGY \arrow[u, "\xi_F;1_{FGY}"] \arrow[ur, "i_{I_{\mathcal{D}}};1_{FGY}"'] \arrow[ddl, "\lambda_{\mathcal{D}_{FGY}}"'] & & \\
& Y \arrow[dl, "i_Y"] & I_{\mathcal{D}}; Y \arrow[l, "\lambda_{\mathcal{D}_Y}"] \arrow[ul, "1_{I_{\mathcal{D}}};i_Y"'] \arrow[uu, "i_{I_{\mathcal{D}}};i_Y"'] \arrow[dr, "i_{I_{\mathcal{D}};Y}"] & \\
FGY & & & FG(I_{\mathcal{D}}; Y), \arrow[lll, "FG(\lambda_{\mathcal{D}_Y})"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(GY; I_{\mathcal{C}}) \arrow[rrr, "F(1_{GY};\xi_G)"] \arrow[dddd, "F(\rho_{\mathcal{C}_{GY}})"'] & & & F(GY; GI_{\mathcal{D}}) \arrow[dddd, "F\nabla_{G_Y,I_{\mathcal{D}}}"] \\
& FGY; F(I_{\mathcal{C}}) \arrow[ul, "\nabla_{F_{GY,I_{\mathcal{C}}}}"'] \arrow[r, "1_{FGY};F(\xi_G)"] & FGY; FGI_{\mathcal{D}} \arrow[ur, "\nabla_{F_{GY,GI_{\mathcal{D}}}}"] & \\
& FGY; I_{\mathcal{D}} \arrow[u, "1_{FGY};\xi_F"] \arrow[ur, "1_{FGY};i_{I_{\mathcal{D}}}"'] \arrow[ddl, "\rho_{\mathcal{D}_{FGY}}"'] & & \\
& Y \arrow[dl, "i_Y"] & Y; I_{\mathcal{D}} \arrow[l, "\rho_{\mathcal{D}_Y}"] \arrow[ul, "i_Y;1_{I_{\mathcal{D}}}"'] \arrow[uu, "i_Y;i_{I_{\mathcal{D}}}"'] \arrow[dr, "i_{Y;I_{\mathcal{D}}}"] & \\
FGY & & & FG(Y; I_{\mathcal{D}}). \arrow[lll, "FG(\rho_{\mathcal{D}_Y})"]
\end{tikzcd}
$$

where we write ";" instead of $\otimes$ for simplicity, which yields unitality of G.

For G to be a symmetric monoidal functor, we still need to prove that for every couple of objects Y_1, Y_2 in $\mathcal{D}$, the diagram

$$
\begin{tikzcd}
GY_1 \otimes GY_2 \arrow[r, "B_{\mathcal{C}_{GY_1,GY_2}}"] \arrow[d, "\nabla_{G_{Y_1,Y_2}}"'] & GY_2 \otimes GY_1 \arrow[d, "\nabla_{G_{Y_2,Y_1}}"] \\
G(Y_1 \otimes Y_2) \arrow[r, "GB_{\mathcal{D}_{Y_1,Y_2}}"'] & G(Y_2 \otimes Y_1)
\end{tikzcd}
$$

commutes. Again, this is enough to prove this for the image under F. Note that the following big diagram commutes,

$$
\begin{tikzcd}
F(GY_1 \otimes GY_2) \arrow[r, "FB_{\mathcal{C}_{GY_1,GY_2}}"] \arrow[ddd, bend right=60, "F\nabla_{G_{Y_1,Y_2}}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(GY_2 \otimes GY_1) \arrow[ddd, bend left=60, "F\nabla_{G_{Y_2,Y_1}}"] \\
FGY_1 \otimes FGY2 \arrow[u, "\nabla_{F_{GY_1,GY_2}}"] \arrow[r, "B_{\mathcal{D}_{FGY_1,FGY_2}}"] \arrow[dr, phantom, "\circlearrowleft"] & FGY_2 \otimes FGY_1 \arrow[u, "\nabla_{F_{GY_2,GY_1}}"'] \\
Y_1 \otimes Y_2 \arrow[u, "i_{Y_1} \otimes i_{Y_2}"] \arrow[r, "B_{\mathcal{D}_{Y_1,Y_2}}"] \arrow[d, "i_{Y_1 \otimes Y_2}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y_2 \otimes Y_1 \arrow[u, "i_{Y_2} \otimes i_{Y_1}"'] \arrow[d, "i_{Y_2 \otimes Y_1}"] \\
FG(Y_1 \otimes Y_2) \arrow[r, "FGB_{\mathcal{D}_{Y_1,Y_2}}"'] & FG(Y_2 \otimes Y_1).
\end{tikzcd}
$$

Since the upper diagram commutes by naturality of ∇_F, the middle one by naturality of the braiding $B_{\mathcal{D}}$, the lower one by naturality of i, and the vertical compositions are equal to $F\nabla_G$ by monoidality of i. This yields that G is a strong symmetric monoidal functor.

We still need to prove that G preserves comonoids, i.e., that for all objects Y in $\mathcal{D}$, the diagram

$$\begin{tikzcd}
& GY \arrow[dl, "copy_{GY}"'] \arrow[dr, "G(copy_Y)"] & \\
GY \otimes GY \arrow[rr, "\nabla_{G_{Y,Y}}"'] & & G(Y \otimes Y),
\end{tikzcd}$$

commutes. We will prove it for the image under F. Since i is natural and monoidal, with deterministic components, the following diagram commutes,

$$\begin{tikzcd}
FGY \arrow[rr, "FG(copy_Y)"] \arrow[ddd, "copy_{FGY}"'] & & FG(Y \otimes Y) \\
& Y \arrow[ul, "i_Y"] \arrow[d, "copy_Y"'] & \\
& Y \otimes Y \arrow[uur, "i_{Y\otimes Y}"] \arrow[dl, "i_Y \otimes i_Y"] & \\
FGY \otimes FGY \arrow[rr, "\nabla_{F_{GY,GY}}"'] & & F(GY \otimes GY), \arrow[uuu, "F\nabla_{G_{Y,Y}}"']
\end{tikzcd}$$

and the diagram

$$\begin{tikzcd}
& FGY \arrow[ddl, bend right, "F(copy_{GY})"'] \arrow[ddr, bend left, "FG(copy_Y)"] \arrow[d, "copy_{FGY}"] & \\
& FGY \otimes FGY \arrow[dl, "\nabla_{F_{GY,GY}}"] & \\
F(GY \otimes GY) \arrow[rr, "F\nabla_{G_{Y,Y}}"'] & & FG(Y \otimes Y),
\end{tikzcd}$$

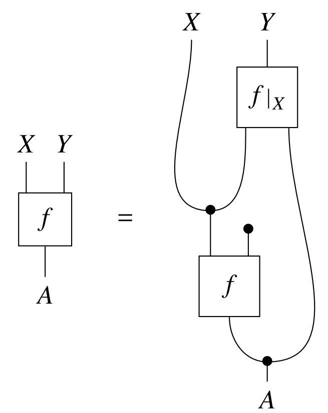
commutes since F preserves comonoids for the left hand side of the diagram, and the right hand side is exactly the diagram above. This yields that G is a 1-morphism in *Markov*. We have then proven that $F : \mathcal{C} \to \mathcal{D}$ is a comonoid equivalence.

□

3.3 Conditionals

In statistics, we often use what is called *conditional statistics*. If we have a joint probability $p(x, y|a)$, we can find conditional probabilities $p(x|a, y)$ and $p(y|a, x)$, i.e., if we know the probability of x and y together, we can compute the probability of x knowing y and the probability of y knowing x. In a Markov category, it is not evident that we can compute such conditionals. In this section, we will give a categorical definition of conditionals, by T. Fritz, and then see what it means in our examples.

Definition 3.11. [2, 11.5]. Let $\mathcal{C}$ be a Markov category. We say that $\mathcal{C}$ *has conditionals* if for every morphism $f : A \to X \otimes Y$, there is $f|_X : X \otimes A \to Y$ such that



, (27)

i.e.,

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[dl, "copy_A"'] & X \otimes Y & \\
A \otimes A \arrow[d, "f \otimes 1_A"'] & & & \\
X \otimes Y \otimes A \arrow[dr, "1_X \otimes del_Y \otimes 1_A"'] & & & X \otimes X \otimes A, \arrow[uul, "1_X \otimes f|_X"'] \\
& X \otimes I \otimes A \arrow[r, "\rho_X \otimes 1_A"'] & X \otimes A \arrow[ur, "copy_X \otimes 1_A"'] &
\end{tikzcd}$$

commutes.

Example 3.12. *FinStoch* has conditionals. For all morphisms $f : A \to X \times Y$ in *FinStoch*, let $g_{x,a} : Y \to [0, 1]$ be such that

$$\sum_{y \in Y} g_{x,a}(y) = 1.$$

for all $x \in X$, $a \in A$. Then define $f|_X : X \times A \to Y$ by the usual notion of conditionals in statistics.

$$f|_X(y|x, a) = \begin{cases} \frac{f(x,y|a)}{\sum_{y_1 \in Y} f(x,y_1|a)}, & \text{if } \sum_{y_1 \in Y} f(x, y_1|a) > 0, \\ g_{x,a}(y), & \text{if } \sum_{y_1 \in Y} f(x, y_1|a) = 0, \end{cases}$$

for all a in A, x in X and y in Y. We prove that $f|_X : X \times A \to Y$ satisfies (27). For all x in X, y in Y and a in A, the right side of (27) becomes in *FinStoch*

$$\begin{aligned}
&(1_X \otimes f|_X) \circ (copy_X \otimes 1_A) \circ (\rho_X \otimes 1_A) \circ (1_X \otimes del_Y \otimes 1_A) \circ (f \otimes 1_A) \circ copy_A(x, y|a) \\
=&(1_X \otimes f|_X) \circ (copy_X \circ \rho_X \circ (1_X \otimes del_Y) \circ f) \otimes 1_A) \circ copy_A(x, y|a) \\
=&(1_X \otimes f|_X) \circ (copy_X \circ \rho_X \circ (1_X \otimes del_Y) \circ f) \otimes 1_A)(x, y|a, a) \\
=&\sum_{\substack{x_1 \in X, \\ x_2 \in X, \\ a_1 \in A}} (1_X \otimes f|_X)(x, y|x_1, x_2, a_1) \cdot (copy_X \circ \rho_X \circ (1_X \otimes del_Y) \circ f) \otimes 1_A)(x, y|a, a) \\
=&\sum_{\substack{x_1 \in X, \\ x_2 \in X, \\ a_1 \in A}} \delta_{x,x_1} \cdot f|_X(y|x_2, a_1) \sum_{\substack{x_3 \in X, \\ (x_4,0) \in X \times I, \\ (x_5,y_1) \in X \times Y,}} copy_X(x_1, x_2|x_3) \cdot \rho_X(x_3|x_4, 0) \cdot \delta_{x_4,x_5} \cdot del_Y(0|y_1) \cdot f(x_5, y_1|a) \cdot \delta_{a_1,a} \qquad (28) \\
=&\sum_{x_2 \in X} f|_X(y|x_2, a) \sum_{\substack{x_3 \in X, \\ (x_5,y_1) \in X \times Y,}} \delta_{x,x_2,x_3} \cdot \delta_{x_3,x_5} \cdot f(x_5, y_1|a) \\
=&f|_X(y|x, a) \sum_{y_1 \in Y} f(x, y_1|a).
\end{aligned}$$

If $\sum_{y_1 \in Y} f(x, y_1|a) > 0$,

$$f|_X(y|x, a) \sum_{y_1 \in Y} f(x, y_1|a) = \frac{f(x, y|a)}{\sum_{y_1 \in Y} f(x, y_1|a)} \sum_{y_1 \in Y} f(x, y_1|a) = f(x, y|a),$$

and if $\sum_{y_1 \in Y} f(x, y_1|a) = 0$,

$$f|_X(y|x, a) \sum_{y_1 \in Y} f(x, y_1|a) = 0 = f(x, y|a),$$

and $f|_X : X \times A \to Y$ satisfies (27), establishing that *FinStoch* has conditionals.

If $\tilde{f} : A \to X \times Y$ is a deterministic morphism in *FinStoch*, we show that we can define $\tilde{f}|_X : X \times A \to Y$ such that it is a deterministic morphism. There exists $f : A \to X \times Y$ in *Set* such that for all a in A, x in X and y in Y,

$$\tilde{f}(x, y|a) = \delta_{x,f_X(a)} \delta_{y,f_Y(a)}.$$

where f_X and f_Y are the marginalisations of f on X and on Y, respectively. Let $g_{x,a} : Y \to [0, 1]$ be defined as follows.

$$g_{x,a}(y) := \delta_{y,f_Y(a)}, \quad \text{for all } y \in Y,$$

since

$$\sum_{y \in Y} g_{x,a}(y) = 1,$$

we can define $\tilde{f}|_X : X \times A \to Y$ as follows. For all a in A, x in X and y in Y,

$$\begin{aligned}
\tilde{f}|_X(y|x,a) &= \begin{cases} \frac{f(x,y|a)}{\sum_{y_1 \in Y} f(x,y_1|a)}, & \text{if } \sum_{y_1 \in Y} f(x,y_1|a) > 0, \\ g_{x,a}(y), & \text{if } \sum_{y_1 \in Y} f(x,y_1|a) = 0, \end{cases} \\
&= \begin{cases} \frac{\delta_{x,f_X(a)} \delta_{y,f_Y(a)}}{\sum_{y_1 \in Y} \delta_{x,f_X(a)} \delta_{y_1,f_Y(a)}}, & \text{if } \sum_{y_1 \in Y} \delta_{x,f_X(a)} \delta_{y_1,f_Y(a)} > 0, \\ \delta_{y,f_Y(a)}, & \text{if } \sum_{y_1 \in Y} \delta_{x,f_X(a)} \delta_{y_1,f_Y(a)} = 0. \end{cases}
\end{aligned}$$

For all a in A, x in X and y in Y,

- if $x = f_X(a)$, there exists a unique $y_1 \in Y$ such that $y_1 = f_Y(a)$ and

  $$\sum_{y_1 \in Y} \delta_{x,f_X(a)} \cdot \delta_{y_1,f_Y(a)} = 1,$$

  so that

  $$\tilde{f}|_X(y|x,a) = \frac{\delta_{x,f_X(a)} \cdot \delta_{y,f_Y(a)}}{\sum_{y_1 \in Y} \delta_{x,f_X(a)} \cdot \delta_{y_1,f_Y(a)}} = \delta_{y,f_Y(a)}.$$

- if $x \neq f_X(a)$,

  $$\sum_{y_1 \in Y} \delta_{x,f_X(a)} \cdot \delta_{y_1,f_Y(a)} = 0,$$

  so that

  $$\tilde{f}|_X(y|x,a) = g_{x,a}(y) = \delta_{y,f_Y(a)}.$$

We finally have

$$\tilde{f}|_X(y|x,a) = \delta_{y,f_Y(a)}.$$

i.e., $\tilde{f}|_X : X \times A \to Y$ is deterministic. This comes from a more general statement, that we will see below.

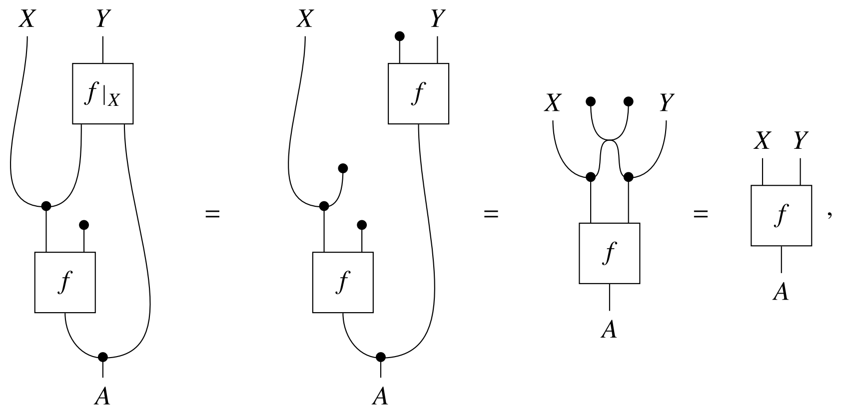
Proposition 3.13. *Let C be a Markov category. Then C_{det} has conditionals.*

Proof. For all morphism $f : A \to X \otimes Y$ in C_{det}, we can define $f|_X : X \times A \to Y$ as follows.

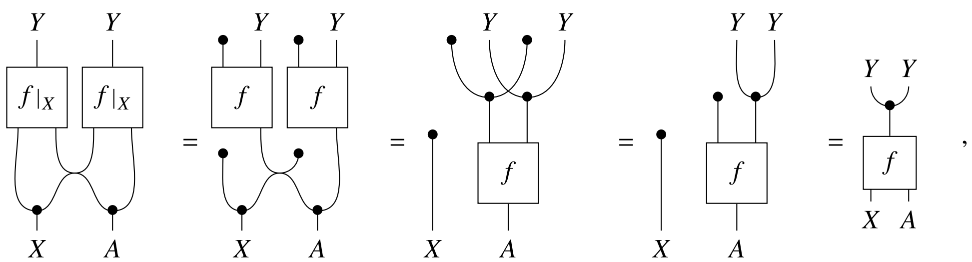
$$f|_X := del_X \otimes ((del_X \otimes 1_Y) \circ f) = del_X \otimes f_Y, \tag{29}$$

with $f_Y := (del_X \otimes 1_Y) \circ f$ the marginalisation of f on Y.

We show that $f|_X$ satisfies (27) and that it is a deterministic morphism.



where the first equality holds by definition (29), the second by determinism of f and the third by counitality (12). This establishes (27), and $f|_X : X \times A \to Y$ is deterministic since



where the first and last equalities hold by definition of $f|_X$ (29), the second by counitality (12) and determinism of f, and the third by counitality (12). This proves that C_{det} has conditionals. □

Example 3.14. Since *Set* is the category of deterministic morphisms of *FinStoch*, it has conditionals.

We want to see in what extend categories with conditionals are preserved. We will analyse the case of the category of functors between an arbitrary category, and a Markov category with conditionals. We will first see that it is a Markov category, and then explain the open problem whether the category of functors have conditionals or not.

Let C be a Markov category and $\mathcal{D}$ a small category. Then we will write $Fun(\mathcal{D}, C)$ for the category of functors between $\mathcal{D}$ and C, with morphisms the natural transformations between those functors. We use notation $X = (X_d)_{d \in D}$ for such functors.

Lemma 3.15. *Let C be a Markov category, $\mathcal{D}$ a small category and a functor $F : C^n \to C$, with n in $\mathbb{N}$. Let $\tilde{F} : Fun(\mathcal{D}, C)^n \to Fun(\mathcal{D}, C)$ be defined by the composition,*

$$Fun(\mathcal{D}, C)^n \xrightarrow{\Phi} Fun(\mathcal{D}^n, C^n) \xrightarrow{\Psi} Fun(\mathcal{D}, C^n) \xrightarrow{F_*} Fun(\mathcal{D}, C),$$

where Φ, Ψ and F_* are defined as follows.

- – *For all objects $X = (X^i)_{0<i\leq n}$ in $Fun(\mathcal{D}, C)^n$,*

  $$\Phi((X^i)_{0<i\leq n}) : \mathcal{D}^n \longrightarrow C^n,$$

  $$(d_i)_{0<i\leq n} \longmapsto (X^i(d_i))_{0<i\leq n}.$$

  – *For all sets of natural transformations $\tau = (\tau^i)_{0<i\leq n} : X = (X^i)_{0<i\leq n} \to Y = (Y_i)_{0<i\leq n}$ with $\tau^i : X^i \to Y^i$, morphism in $Fun(\mathcal{D}, C)^n$,*

  $$\Phi(\tau) : \Phi((X^i)_{0<i\leq n}) \Longrightarrow \Phi((Y^i)_{0<i\leq n}),$$

  $$\Phi(\tau)_{(d_i)_{0<i\leq n}} := (\tau^i_{d_i})_{0<i\leq n} : (X^i(d_i))_{0<i\leq n} \longrightarrow (Y^i(d_i))_{0<i\leq n},$$

  for all $(d_i)_{0<i\leq n}$ in $\mathcal{D}^n$.

- – *For all objects $G : \mathcal{D}^n \to C^n$ in $Fun(\mathcal{D}^n, C^n)$,*

  $$\Psi(G) := G \circ \Delta^{(n)},$$

  where $\Delta^{(n)} : \mathcal{D} \to \mathcal{D}^n$ is the iterated diagonal, i.e., $\Delta^{(n)}(d) = (d_i = d)_{0<i\leq n}$ for all objects d in $\mathcal{D}$.

  – *For all morphisms $\sigma : G \Rightarrow H$ in $Fun(\mathcal{D}^n, C^n)$,*

  $$\Psi(\sigma) := \sigma * \Delta^{(n)},$$

  i.e., for all d in $\mathcal{D}$,

  $$(\Psi(\sigma)^i_d)_{0<i\leq n} := (\sigma^i_d)_{0<i\leq n}.$$

- *F_* is the postcomposition functor with F.*

then $\tilde{F} : Fun(\mathcal{D}, C)^n \to Fun(\mathcal{D}, C)$ is a functor.

If, moreover, there exists a second functor $G : C^n \to C$ and a natural transformation $\sigma : F \Rightarrow G$, define $\Sigma : \tilde{F} \Rightarrow \tilde{G}$ as follows. For all object $X = (X^i)_{0<i\leq n}$ in $Fun(\mathcal{D}, C)^n$, the natural transformations $\Sigma_X : \tilde{F}(X) \Rightarrow \tilde{G}(X)$ are defined pointwise by

$$(\Sigma_X)_d := \sigma_{X_d},$$

then $\Sigma : \tilde{F} \Rightarrow \tilde{G}$ is a natural transformation.

Proof. • *We show that $\tilde{F}$ is a functor.*

- *We show that Φ is a functor*. For all objects $X = (X^i)_{0<i\leq n}$ in $Fun(\mathcal{D},C)^n$, for all $(d_i)_{0<i\leq n}$ in $\mathcal{D}^n$,

$$\Phi(Id_X)_{(d_i)_{0<i\leq n}} = (Id_{X^i(d_i)})_{0<i\leq n},$$

and $\Phi(Id_X) = Id_{\Phi_X}$. Now for all couples of sets of natural transformations $\tau_1 = (\tau_1^i)_{0<i\leq n} : X = (X^i)_{0<i\leq n} \to Y = (Y_i)_{0<i\leq n}$, $\tau_2 = (\tau_2^i)_{0<i\leq n} : Y = (Y^i)_{0<i\leq n} \to Z = (Z_i)_{0<i\leq n}$ with $\tau_1^i : X^i \to Y^i$ and $\tau_2^i : Y^i \to Z^i$, morphisms in $Fun(\mathcal{D},C)^n$, for all $(d_i)_{0<i\leq n}$ in $\mathcal{D}^n$,

$$\Phi(\tau_2 \circ \tau_1)_{(d_i)_{0<i\leq n}} = ((\tau_2 \circ \tau_1)^i_{d_i})_{0<i\leq n} = ((\tau_2^i)_{d_i} \circ (\tau_1^i)_{d_i})_{0<i\leq n} = (\Phi(\tau_2) \circ \Phi(\tau_1))_{(d_i)_{0<i\leq n}},$$

this proves that Φ is a functor.

- *We show that Ψ is a functor.* For all object $G : \mathcal{D}^n \to C^n$, for all d in $\mathcal{D}$,

$$(\Psi(Id_G))_d = (Id_G * \Delta^{(n)})_d = (Id_{G(d)})_{0\leq i\leq n},$$

and $\Psi(Id_G) = Id_{\Psi(G)}$. Now for all couples of natural transformations $\sigma_1 : G \Rightarrow H$, $\sigma_1 : H \Rightarrow E$, morphisms in $Fun(\mathcal{D}^n, C^n)$,

$$\Psi(\sigma_2 \circ \sigma_1) = (\sigma_2 \circ \sigma_1) * \Delta^{(n)} = (\sigma_2 * \Delta^{(n)}) \circ (\sigma_1 * \Delta^{(n)}) = \Psi(\sigma_2) \circ \Psi(\sigma_1),$$

this proves that Ψ is a functor.

- *We prove that F^* is a functor.* For all objects $G : \mathcal{D} \to C^n$ in $Fun(\mathcal{D}, C^n)$, for all objects d in $\mathcal{D}$,

$$(F^*(Id_G))_d = F(Id_{G(d)}) = Id_{F\circ G(d)},$$

by functoriality of F, and so $F^*(Id_G) = Id_{F^*(G)}$. Now for all couples of natural transformations $\eta_1 : G \Rightarrow H$, $\eta_1 : H \Rightarrow E$, morphisms in $Fun(\mathcal{D}, C^n)$, and for all d in $\mathcal{D}$,

$$(F^*(\eta_2 \circ \eta_1)) = F((\eta_2 \circ \eta_1)_d) = F((\eta_2)_d \circ (\eta_1)_d),$$

this proves that F^* is a functor.

Since $\tilde{F}$ is the composition of three functors, it is a functor.

- *We now show that Σ is a natural transformation*. For all set of natural transformations $\tau = (\tau^i)_{0<i\leq n} : X = (X^i)_{0<i\leq n} \to Y = (Y_i)_{0<i\leq n}$ with $\tau^i : X^i \to Y^i$, morphism in $Fun(\mathcal{D},C)^n$,

$$\begin{tikzcd} \tilde{F}(X)(d) \arrow[r, "F(\tau_d)"] \arrow[d, "\sigma_{X_d}"'] & \tilde{F}(Y)(d) \arrow[d, "\sigma_{Y_d}"] \\ \tilde{G}(X)(d) \arrow[r, "G(\tau_d)"'] & \tilde{G}(Y)(d), \end{tikzcd}$$

commutes for all d in $\mathcal{D}$ by naturality of σ, which makes the following two diagrams commute,

$$\begin{tikzcd} \tilde{F}(X)(d) \arrow[r, "\tilde{F}(\tau)_d"] \arrow[d, "(\Sigma_X)_d"'] & \tilde{F}(Y)(d) \arrow[d, "(\Sigma_Y)_d"] \\ \tilde{G}(X)(d) \arrow[r, "\tilde{G}(\tau)_d"'] & \tilde{G}(Y)(d), \end{tikzcd}$$

$$\begin{tikzcd} \tilde{F}(X) \arrow[r, "\tilde{F}(\tau)"] \arrow[d, "\Sigma_X"'] & \tilde{F}(Y) \arrow[d, "\Sigma_Y"] \\ \tilde{G}(X) \arrow[r, "\tilde{G}(\tau)"'] & \tilde{G}(Y), \end{tikzcd}$$

and this establishes naturality of Σ.

□

Proposition 3.16. *[2, 7.1.]. Let $\mathcal{C}$ be a Markov category and $\mathcal{D}$ a small category. Then we can endow $Fun(\mathcal{D}, \mathcal{C})$ with the structure of a Markov category. The tensor product will be defined as follows. For two objects X, Y in $Fun(\mathcal{D}, \mathcal{C})$,*

$$(X \otimes Y)_d := X_d \otimes_{\mathcal{C}} Y_d, \quad \forall d \in Ob\mathcal{D},$$

and for two natural transformations $\tau : A \Rightarrow X$, $\sigma : B \Rightarrow Y$, morphisms in $Fun(\mathcal{D}, \mathcal{C})$,

$$(\tau \otimes \sigma)_d := \tau_d \otimes_{\mathcal{C}} \sigma_d : (A \otimes B)_d \to (X \otimes Y)_d.$$

The comultiplications will also be defined pointwise:

$$(copy_X)_d := copy_{\mathcal{C}_{X_d}},$$

for all d in $\mathcal{D}$ and the counits will be defined by:

$$(del_X)_d := del_{\mathcal{C}_{X_d}},$$

for all d in $\mathcal{D}$

Proof.

- *We first show that $Fun(\mathcal{D}, \mathcal{C})$ is a monoidal category.*

1. The tensor product defined in the statement is a bifunctor since we can apply the first part of Lemma 3.15 to the bifunctor $- \otimes_{\mathcal{C}} -$, which gives the definition of the tensor product in $Fun(\mathcal{D}, \mathcal{C})$ from the statement of the Proposition.
2. The unit object is the constant functor $I = (I_d)_{d \in \mathcal{D}}$, with $I_d = I_{\mathcal{C}}$ for all d in $\mathcal{D}$.
3. For all triples of functors $X, Y, Z : \mathcal{D} \to \mathcal{C}$, objects in $Fun(\mathcal{D}, \mathcal{C})$, we define $a_{X,Y,Z} : (X \otimes Y) \otimes Z \Rightarrow X \otimes (Y \otimes Z)$ pointwise:
$$(a_{X,Y,Z})_d := a_{\mathcal{C}_{X_d,Y_d,Z_d}},$$
this defines a natural isomorphism $a : (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)$ by Lemma 3.15.
4. For all functors $X : \mathcal{C} \to \mathcal{D}$, object in $Fun(\mathcal{D}, \mathcal{C})$, we define the left and right unitors pointwise :
$$(\lambda_X)_d := \lambda_{\mathcal{C}_{X_d}},$$
$$(\rho_X)_d := \rho_{\mathcal{C}_{X_d}},$$
this defines natural isomorphisms $\lambda : I \otimes - \Rightarrow 1$ and $\rho : - \otimes I \Rightarrow 1$ by Lemma 3.15.
5. We show the pentagon identity in $Fun(\mathcal{D}, \mathcal{C})$. For all quadruples of functors $W, X, Y, Z : \mathcal{D} \to \mathcal{C}$, objects in $Fun(\mathcal{D}, \mathcal{C})$,
$$\begin{tikzcd}
& (W_d \otimes X_d) \otimes (Y_d \otimes Z_d) \arrow[dr, "a_{\mathcal{C}_{W_d,X_d,Y_d \otimes Z_d}}"] & \\
((W_d \otimes X_d) \otimes Y_d) \otimes Z_d \arrow[ur, "a_{\mathcal{C}_{W \otimes X,Y,Z}}"] \arrow[d, "a_{\mathcal{C}_{W_d,X_d,Y_d}} \otimes 1_{Z_d}"'] & & W_d \otimes (X_d \otimes (Y_d \otimes Z_d)) \\
(W_d \otimes (X_d \otimes Y_d)) \otimes Z_d \arrow[rr, "a_{\mathcal{C}_{W_d,X_d \otimes Y_d,Z_d}}"'] & & W_d \otimes ((X_d \otimes Y_d) \otimes Z_d), \arrow[u, "1_{W_d} \otimes a_{\mathcal{C}_{X_d,Y_d,Z_d}}"']
\end{tikzcd}$$
commutes for all objects d in $\mathcal{D}$ since $\mathcal{D}$ is monoidal, making the two following diagram commute:
$$\begin{tikzcd}
& ((W \otimes X) \otimes (Y \otimes Z))_d \arrow[dr, "(a_{W,X,Y \otimes Z})_d"] & \\
(((W \otimes X) \otimes Y) \otimes Z)_d \arrow[ur, "(a_{W \otimes X,Y,Z})_d"] \arrow[d, "(a_{W,X,Y} \otimes 1_Z)_d"'] & & (W \otimes (X \otimes (Y \otimes Z)))_d \\
((W \otimes (X \otimes Y)) \otimes Z)_d \arrow[rr, "(a_{W,X \otimes Y,Z})_d"'] & & (W \otimes ((X \otimes Y) \otimes Z))_d, \arrow[u, "(1_W \otimes a_{X,Y,Z})_d"']
\end{tikzcd}$$
$$\begin{tikzcd}
& (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "a_{W,X,Y \otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "a_{W \otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y} \otimes 1_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{W,X \otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z), \arrow[u, "1_W \otimes a_{X,Y,Z}"']
\end{tikzcd}$$
establishing the pentagon identity.

6. We show the triangle identity in $Fun(\mathcal{D}, C)$. For all functors $X, Y : \mathcal{D} \to C$, objects in $Fun(\mathcal{D}, C)$,

$$\begin{tikzcd}
(X_d \otimes_C I_C) \otimes_C Y_d \arrow[rr, "a_{C_{X_d, I_d, Y_d}}"] \arrow[dr, "\rho_{C_{X_d}} \otimes 1_{Y_d}"'] & & X_d \otimes_C (I_C \otimes_C Y_d), \arrow[dl, "1_{X_d} \otimes_C \lambda_{C_{Y_d}}"] \\
& X_d \otimes_C Y_d &
\end{tikzcd}$$

commutes for all object d in $\mathcal{D}$ since C is a monoidal category, making the two following diagrams commute:

$$\begin{tikzcd}
((X \otimes I) \otimes Y)_d \arrow[r, "(a_{X,I,Y})_d"] \arrow[d, "(\rho_X \otimes 1_Y)_d"'] & (X \otimes (I \otimes Y))_d, \arrow[dl, "(1_X \otimes \lambda_Y)_d"] \\
(X \otimes Y)_d &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[r, "a_{X,I,Y}"] \arrow[d, "\rho_X \otimes 1_Y"'] & X \otimes (I \otimes Y), \arrow[dl, "1_X \otimes \lambda_Y"] \\
X \otimes Y &
\end{tikzcd}$$

establishing the triangle identity.

This makes $Fun(\mathcal{D}, C)$ into a monoidal category.

- *We show that $Fun(\mathcal{D}, C)$ is symmetric.* For all couples of functors $X, Y : \mathcal{D} \to C$, objects in $Fun(\mathcal{D}, C)$, we define $B_{X,Y} : X \otimes Y \Rightarrow Y \otimes X$ pointwise:

$$(B_{X,Y})_d := B_{\mathcal{D}_{X_d, Y_d}},$$

this defines a natural isomorphism $B_{-,\cdot} : - \otimes \cdot \Rightarrow \cdot \otimes -$ by Lemma 3.15.

  - We show the first hexagon identity in $Fun(\mathcal{D}, C)$. For all triples of functors $X, Y, Z : \mathcal{D} \to C$, objects in $Fun(\mathcal{D}, C)$, the following diagram commutes

$$\begin{tikzcd}
(X_d \otimes Y_d) \otimes Z_d \arrow[r, "a_{C_{X_d, Y_d, Z_d}}"] \arrow[d, "B_{C_{X_d, Y_d}} \otimes 1_{Z_d}"'] & X_d \otimes (Y_d \otimes Z_d) \arrow[r, "B_{C_{X, Y \otimes Z}}"] & (Y_d \otimes Z_d) \otimes X_d \arrow[d, "a_{C_{Y_d, Z_d, X_d}}"] \\
(Y_d \otimes X_d) \otimes Z_d \arrow[r, "a_{C_{Y,X,Z}}"'] & Y_d \otimes (X_d \otimes Z_d) \arrow[r, "1_{Y_d} \otimes B_{C_{X_d, Z_d}}"'] & Y_d \otimes (Z_d \otimes X_d),
\end{tikzcd}$$

for all objects d in $\mathcal{D}$ by the hexagon identity in the symmetric monoidal category C, making the following diagram commute:

$$\begin{tikzcd}
((X \otimes Y) \otimes Z)_d \arrow[r, "(a_{X,Y,Z})_d"] \arrow[d, "(B_{X,Y} \otimes 1_Z)_d"'] & (X \otimes (Y \otimes Z))_d \arrow[r, "(B_{X, Y \otimes Z})_d"] & ((Y \otimes Z) \otimes X)_d \arrow[d, "(a_{Y,Z,X})_d"] \\
((Y \otimes X) \otimes Z)_d \arrow[r, "(a_{Y,X,Z})_d"'] & (Y \otimes (X \otimes Z))_d \arrow[r, "(1_Y \otimes B_{X,Z})_d"'] & (Y \otimes (Z \otimes X))_d,
\end{tikzcd}$$

for all objects d in $\mathcal{D}$, and

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "B_{X,Y} \otimes 1_Z"'] & X \otimes (Y \otimes Z) \arrow[r, "B_{X, Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "a_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[r, "1_Y \otimes B_{X,Z}"'] & Y \otimes (Z \otimes X),
\end{tikzcd}$$

commutes. This establishes first hexagon identity.

  - We show unit coherence and symmetry axiom in $Fun(\mathcal{D}, C)$. For all functors $X : \mathcal{D} \to C$, objects in $Fun(\mathcal{D}, C)$, the following diagrams commute for all objects d in $\mathcal{D}$ by unit coherence and symmetry axiom in the symmetric monoidal category C,

$$\begin{tikzcd}
X_d \otimes_C I_C \arrow[r, "B_{C_{X_d, I_C}}"] \arrow[dr, "\rho_{C_{X_d}}"'] & I_C \otimes_C X_d, \arrow[d, "\lambda_{C_{X_d}}"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
X_d \otimes_C Y_d \arrow[r, "B_{C_{X_d, Y_d}}"] \arrow[dr, equal] & Y_d \otimes_C X_d, \arrow[d, "B_{C_{Y_d, X_d}}"] \\
& X_d \otimes_C Y_d
\end{tikzcd}$$

implying that the following diagrams commute for all object d in $\mathcal{D}$,

$$\begin{tikzcd}
(X \otimes I)_d \arrow[r, "(B_{X,I})_d"] \arrow[dr, "(\rho_X)_d"'] & (I \otimes X)_d, \arrow[d, "(\lambda_X)_d"] \\
& X_d
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y)_d \arrow[r, "(B_{X,Y})_d"] \arrow[dr, equal] & (Y \otimes X)_d, \arrow[d, "(B_{Y,X})_d"] \\
& (X \otimes Y)_d
\end{tikzcd}$$

and this last two diagrams commute,

$$\begin{tikzcd}
X \otimes I \arrow[r, "B_{X,I}"] \arrow[dr, "\rho_X"'] & I \otimes X, \arrow[d, "\lambda_X"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "B_{X,Y}"] \arrow[dr, equal] & Y \otimes X, \arrow[d, "B_{Y,X}"] \\
& X \otimes Y
\end{tikzcd}$$

establihing unit coherence and symmetry axiom in $Fun(\mathcal{D}, \mathcal{C})$.

- *The symmetric monoidal category $Fun(\mathcal{D}, \mathcal{C})$, endowed with the structure of the statement, is a Markov category.*
  - We verify the commutative comonoid equations in $Fun(\mathcal{D}, \mathcal{C})$. For all functors $X : \mathcal{D} \to \mathcal{C}$, objects in $Fun(\mathcal{D}, \mathcal{C})$, the following diagrams commute for all d in $\mathcal{D}$ by coassociativity, and counitality in $\mathcal{C}$:

$$\begin{tikzcd}
X_d \arrow[r, "copy_{\mathcal{C}X_d}"] \arrow[d, "copy_{\mathcal{C}X_d}"'] & X_d \otimes_{\mathcal{C}} X_d \arrow[d, "copy_{\mathcal{C}X_d}"] \\
X_d \otimes_{\mathcal{C}} X_d \arrow[r, "copy_{\mathcal{C}X_d} \otimes_{\mathcal{C}} 1_{X_d}"'] & X_d \otimes_{\mathcal{C}} X_d \otimes_{\mathcal{C}} X_d,
\end{tikzcd}$$

$$\begin{tikzcd}
& X_d \otimes_{\mathcal{C}} X_d \arrow[dl, "del_{\mathcal{C}X_d} \otimes_{\mathcal{C}} 1_{\mathcal{C}X_d}"'] \arrow[dr, "1_{\mathcal{C}X_d} \otimes_{\mathcal{C}} del_{\mathcal{C}X_d}"] & \\
I_{\mathcal{C}} \otimes_{\mathcal{C}} X_d \arrow[d, "\lambda_{\mathcal{C}X_d}"'] & & X_d \otimes_{\mathcal{C}} I_{\mathcal{C}} \arrow[d, "\rho_{\mathcal{C}X_d}"] \\
X_d \arrow[r, equal] & X_d \arrow[r, equal] & X_d,
\end{tikzcd}$$

$$\begin{tikzcd}
X_d \arrow[r, "copy_{\mathcal{C}X_d}"] \arrow[d, "copy_{\mathcal{C}X_d}"'] & X_d \otimes_{\mathcal{C}} X_d, \\
X_d \otimes_{\mathcal{C}} X_d \arrow[ur, "B_{\mathcal{C}X_d, X_d}"'] &
\end{tikzcd}$$

making the following diagrams commute for all d in $\mathcal{D}$:

$$\begin{tikzcd}
X_d \arrow[r, "(copy_X)_d"] \arrow[d, "(1_X \otimes copy_X)_d"'] & (X \otimes X)_d \arrow[d, "(copy_X)_d"] \\
(X \otimes X)_d \arrow[r, "(copy_X \otimes 1_X)_d"'] & (X \otimes X \otimes X)_d,
\end{tikzcd}$$

$$\begin{tikzcd}
& (X \otimes X)_d \arrow[dl, "(del_X \otimes 1_X)_d"'] \arrow[dr, "(1_X \otimes del_X)_d"] & \\
(I \otimes X)_d \arrow[d, "(\lambda_X)_d"'] & & (X \otimes I)_d \arrow[d, "(\rho_X)_d"] \\
X_d \arrow[r, equal] & X_d \arrow[r, equal] & X_d,
\end{tikzcd}$$

$$\begin{tikzcd}
X_d \arrow[r, "(copy_X)_d"] \arrow[d, "(copy_X)_d"'] & (X \otimes X)_d, \\
(X \otimes X)_d \arrow[ur, "(B_{X,X})_d"'] &
\end{tikzcd}$$

and finally, the three following diagrams commute:

$$
\begin{tikzcd}
X \arrow[r, "copy_X"] \arrow[d, "1_X \otimes copy_X"'] & X \otimes X \arrow[d, "copy_X"] \\
X \otimes X \arrow[r, "copy_X \otimes 1_X"'] & X \otimes X \otimes X,
\end{tikzcd}
$$

$$
\begin{tikzcd}
& X \otimes X \arrow[dl, "del_X \otimes 1_X"'] \arrow[dr, "1_X \otimes del_X"] & \\
I \otimes X \arrow[d, "\lambda_X"'] & & X \otimes I \arrow[d, "\rho_X"] \\
X \arrow[r, equal] & X \arrow[r, equal] & X,
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "copy_X"] \arrow[d, "copy_X"'] & X \otimes X, \\
X \otimes X \arrow[ur, "B_{X,X}"'] &
\end{tikzcd}
$$

establishing coassociativity, counitality and cocommutativity in $Fun(\mathcal{D}, C)$.

- We now verify compatibility with the monoidal structure in $Fun(\mathcal{D}, C)$. For all couples of functors $X, Y : \mathcal{D} \to C$, objects in $Fun(\mathcal{D}, C)$, the following diagrams commute for all d in $\mathcal{D}$ by compatibility with the monoidal structure in C:

$$
\begin{tikzcd}
X_d \otimes_C Y_d \arrow[r, "del_{C_{X_d}} \otimes_C 1_{X_d}"] \arrow[d, equal] & I_C \otimes_C Y_d \arrow[r, "1_{I_C} \otimes_C del_{C_{Y_d}}"] & I_C \otimes_C I_C \arrow[d, "\lambda_{C_{I_C}}"] \\
X_d \otimes Y_d \arrow[rr, "del_{C_{X_d \otimes Y_d}}"'] & & I_C,
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (X_d \otimes Y_d) \otimes (X_d \otimes Y_d) \arrow[dr, "a^{-1}_{C_{X_d \otimes Y_d, X_d, Y_d}}"] & \\
X_d \otimes_C Y_d \arrow[ur, "copy_{C_{X_d \otimes Y_d}}"] \arrow[d, "copy_{X_d} \otimes_C 1_{Y_d}"'] & & ((X_d \otimes_C Y_d) \otimes_C X_d) \otimes_C Y_d \arrow[d, "a_{C_{X,Y,X}} \otimes_C 1_{Y_d}"] \\
(X_d \otimes_C X_d) \otimes_C Y_d \arrow[d, "1_{C_{X_d \otimes X_d}} \otimes_C copy_{C_{Y_d}}"'] & & (X_d \otimes_C (Y_d \otimes_C X_d) \otimes_C Y_d \arrow[d, "(1_X \otimes_C B_{C_{X_d, Y_d}}) \otimes_C 1_{Y_d}"] \\
(X_d \otimes_C X_d) \otimes_C (Y_d \otimes_C Y_d) \arrow[dr, "a^{-1}_{C_{X \otimes X, Y, Y}}"'] & & (X_d \otimes_C (X_d \otimes_C Y_d)) \otimes_C Y_d, \\
& ((X_d \otimes_C X_d) \otimes_C Y_d) \otimes_C Y_d \arrow[ur, "a_{C_{X_d, X_d, Y_d}} \otimes_C 1_{Y_d}"'] &
\end{tikzcd}
$$

making the two following diagrams to commute for all object d in $\mathcal{D}$:

$$
\begin{tikzcd}
(X \otimes Y)_d \arrow[r, "(del_X \otimes 1_X)_d"] \arrow[d, equal] & (I \otimes Y)_d \arrow[r, "(1_I \otimes del_Y)_d"] & (I \otimes I)_d \arrow[d, "(\lambda_I)_d"] \\
(X \otimes Y)_d \arrow[rr, "(del_{X \otimes Y})_d"'] & & I_d,
\end{tikzcd}
$$

$$
\begin{tikzcd}
& ((X \otimes Y) \otimes (X \otimes Y))_d \arrow[dr, "(a^{-1}_{X\otimes Y,X,Y})_d"] & \\
(X \otimes Y)_d \arrow[ur, "(copy_{X\otimes Y})_d"] \arrow[d, "(copy_X\otimes 1_Y)_d"'] & & (((X \otimes Y) \otimes X) \otimes Y)_d \arrow[d, "(a_{X,Y,X}\otimes 1_Y)_d"] \\
((X \otimes X) \otimes Y)_d \arrow[d, "(1_{X\otimes X}\otimes copy_Y)_d"'] & & ((X \otimes (Y \otimes X) \otimes Y)_d \arrow[d, "(1_X\otimes B_{X,Y})\otimes 1_Y)_d"] \\
((X \otimes X) \otimes (Y \otimes Y))_d \arrow[dr, "(a^{-1}_{X\otimes X,Y,Y})_d"'] & & ((X \otimes (X \otimes Y)) \otimes Y)_d, \\
& (((X \otimes X) \otimes Y) \otimes Y)_d \arrow[ur, "(a_{X,X,Y}\otimes 1_Y)_d"'] &
\end{tikzcd}
$$

and finally the following two diagrams commute,

$$
\begin{tikzcd}
X \otimes Y \arrow[r, "del_X\otimes 1_X"] \arrow[d, equal] & I \otimes Y \arrow[r, "1_I\otimes del_Y"] & I \otimes I \arrow[d, "\lambda_I"] \\
X \otimes Y \arrow[rr, "del_{X\otimes Y}"'] & & I,
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (X \otimes Y) \otimes (X \otimes Y) \arrow[dr, "a^{-1}_{X\otimes Y,X,Y}"] & \\
X \otimes Y \arrow[ur, "copy_{X\otimes Y}"] \arrow[d, "copy_X\otimes 1_Y"'] & & ((X \otimes Y) \otimes X) \otimes Y \arrow[d, "a_{X,Y,X}\otimes 1_Y"] \\
(X \otimes X) \otimes Y \arrow[d, "1_{X\otimes X}\otimes copy_Y"'] & & (X \otimes (Y \otimes X) \otimes Y \arrow[d, "(1_X\otimes B_{X,Y})\otimes 1_Y"] \\
(X \otimes X) \otimes (Y \otimes Y) \arrow[dr, "a^{-1}_{X\otimes X,Y,Y}"'] & & (X \otimes (X \otimes Y)) \otimes Y, \\
& ((X \otimes X) \otimes Y) \otimes Y \arrow[ur, "a_{X,X,Y}\otimes 1_Y"'] &
\end{tikzcd}
$$

establishing compatibility with monoidal structure.

– We finally show naturality of deletion in $Fun(\mathcal{D}, C)$. For every natural transformation $\tau : X \Rightarrow Y$, morphism in $Fun(\mathcal{D}, C)$, the following diagram commutes for all d in $\mathcal{D}$ by naturality of deletion in C (16),

$$
\begin{tikzcd}
X_d \arrow[r, "\tau_d"] \arrow[d, "del_{C_{X_d}}"'] & Y_d, \arrow[dl, "del_{C_{Y_d}}"] \\
I_C &
\end{tikzcd}
$$

making the following diagram commute for all object d in $\mathcal{D}$,

$$
\begin{tikzcd}
X_d \arrow[r, "\tau_d"] \arrow[d, "(del_X)_d"'] & Y_d, \arrow[dl, "(del_Y)_d"] \\
I_d &
\end{tikzcd}
$$

and finally, the following diagram commutes,

$$
\begin{tikzcd}
X \arrow[r, "\tau"] \arrow[d, "del_X"'] & Y, \arrow[dl, "del_Y"] \\
I &
\end{tikzcd}
$$

establishing naturality of deletion in $Fun(\mathcal{D}, C)$.

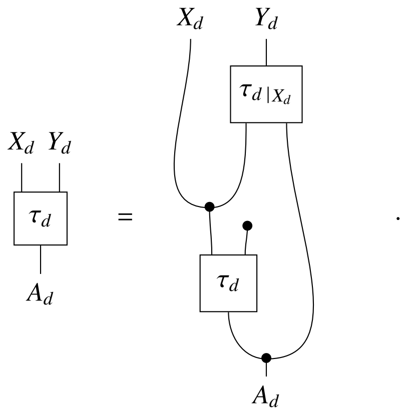
We have proven that $Fun(\mathcal{D}, C)$ is a Markov category.

□

One problem that T. Fritz has raised is the following.

Problem 3.17. *[2, 11.9.]. Suppose that C is a Markov category which has conditionals, and $\mathcal{D}$ a small category. Then does the functor category $Fun(\mathcal{D}, C_{det})$ have conditionals as well ?*

Let us try to answer to this question "naively" and see what problems arise. So if $\tau : A \Rightarrow X \otimes Y$ is a morphism in $Fun(\mathcal{D}, C_{det})$, since C has conditionals, we have that for all object d in $\mathcal{D}$, $\tau_d : A_d \to X_d \otimes Y_d$ is a morphism in C and since C has conditionals, there exists a morphism $\tau_{d\,|X_d} : X_d \otimes A_d \to Y_d$ such that,



So one idea would be to construct $\tau_{|X}$ pointwise by the following.

$$(\tau_{|X})_d := \tau_{d\,|X_d},$$

but then we need to verify that this defines a natural transformation, i.e., that for all morphisms $f : d \to d'$ in $\mathcal{D}$, the following diagram commutes:

$$\begin{tikzcd} X_d \otimes A_d \arrow[r, "\tau_{d\,|X_d}"] \arrow[d, "X(f)\otimes Y(f)"'] & Y_d \arrow[d, "Y(f)"] \\ X_{d'} \otimes A_{d'} \arrow[r, "\tau_{d'\,|X_{d'}}"'] & Y_{d'}, \end{tikzcd}$$

and this is where the problem arises. It is not evident that this diagram commutes. We can think about conditions under which it commutes, or maybe about an other way of defining the conditionals on $Fun(\mathcal{D}, C_{det})$, which we will see in the following example.

Example 3.18. Let $\mathcal{D} = (Two)$ and $C = FinStoch$, with

$$Two = \ 1 \xrightarrow{f} 2,$$

together with the identity morphisms. We want to see if $Fun(\mathcal{D}, C_{det})$ has conditionals. For all natural transformations $\psi : A \to X \times Y$, we have that the diagram

$$\begin{tikzcd} A_1 \arrow[r, "A(f)"] \arrow[d, "\psi_1"'] & A_2 \arrow[d, "\psi_2"] \\ X_1 \times Y_1 \arrow[r, "X(f)\otimes Y(f)"'] & X_2 \times Y_2, \end{tikzcd} \tag{30}$$

commutes. And by Proposition (2.19), the following diagrams commute

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "X(f)\otimes Y(f)"] \arrow[d, "p_{X_1}"'] & X_2 \times Y_2 \arrow[d, "p_{X_2}"] \\ X_1 \arrow[r, "X(f)"'] & X_2, \end{tikzcd} \qquad \begin{tikzcd} X_1 \times X_2 \arrow[r, "X(f)\otimes Y(f)"] \arrow[d, "p_{Y_1}"'] & X_2 \times Y_2 \arrow[d, "p_{Y_2}"] \\ Y_1 \arrow[r, "Y(f)"'] & Y_2. \end{tikzcd}$$

Let us define $\psi_{|X} : X \otimes A \Rightarrow Y$ by

$$(\psi_{|X})_1\,(y_1|x_1, a_1) = \begin{cases} (p_{Y_1} \circ \psi_1)(y_1|a_1), & \text{if } (p_{X_1} \circ \psi_1)(X(f)^{-1}(X(f)(x_1))|a_1) = 0 \\ \frac{\psi_1(X(f)^{-1}(X(f)(x_1)), y_1|a_1)}{(p_{X_1} \circ \psi_1)(X(f)^{-1}(X(f)(x_1))|a_1)}, & \text{if } (p_{X_1} \circ \psi_1)(X(f)^{-1}(X(f)(x_1))|a_1) > 0, \end{cases} \tag{31}$$

where

$$
\begin{aligned}
\psi_1(y_1, X(f)^{-1}(X(f)(x_1))|a_1) &= \sum_{x_1' \in X(f)^{-1}(X(f)(x_1))} \psi_1(x_1', y_1|a_1) = ((X(f)\otimes 1_{Y_1})\circ \psi_1)(X(f)(x_1), y_1|a_1),\\
(p_{X_1}\circ \psi_1)(X(f)^{-1}(X(f)(x_1))|a_1) &= \sum_{x_1' \in X(f)^{-1}(X(f)(x_1))} p_{X_1}\circ \psi_1(x_1'|a_1) = (X(f)\circ p_{X_1}\circ \psi_1)(X(f)(x_1)|a_1),
\end{aligned}
\tag{32}
$$

and

$$
(\psi|_X)_2\,(y_2|x_2,a_2) = \begin{cases} (p_{Y_2}\circ\psi_2)(y_2|a_2), & \text{if } (p_{X_2}\circ\psi_2)(x_2|a_2) = 0,\\ \frac{\psi_2(y_2,x_2|a_2)}{(p_{X_2}\circ\psi_2)(x_2|a_2)}, & \text{if } (p_{X_2}\circ\psi_2)(x_2|a_2) > 0.\end{cases}
$$

We prove that $\psi|_X$ is a natural transformation, i.e., that

$$
\begin{tikzcd}
X_1\times A_1 \arrow[r, "X(f)\otimes A(f)"] \arrow[d, "(\psi|_X)_1"'] & A_2\times X_2 \arrow[d, "(\psi|_X)_2"] \\
Y_1 \arrow[r, "Y(f)"'] & Y_2,
\end{tikzcd}
$$

commutes.

On the one hand, for all a_1 in A_1, x_1 in X_1, and y_2 in Y_2,

$$
\begin{aligned}
&(\psi|_X)_2 \circ (X(f)\otimes A(f))(y_2|x_1,a_1)\\
&= \sum_{(a_2,x_2)\in A_2\times X_2} (\psi|_X)_2\,(y_2|x_2,a_2) X(f)(x_2|x_1) A(f)(a_2|a_1)\\
&= \sum_{(a_2,x_2)\in A_2\times X_2} (\psi|_X)_2\,(y_2|x_2,a_2)\cdot \delta_{X(f)(x_1),x_2}\cdot \delta_{A(f)(a_1),a_2}\\
&= (\psi|_X)_2\,(y_2|X(f)(x_1), A(f)(a_1))\\
&= \begin{cases} (p_{Y_2}\circ\psi_2)(y_2|A(f)(a_1)), & \text{if } (p_{X_2}\circ\psi_2)(X(f)(x_1)|A(f)(a_1)) = 0\\ \frac{\psi_2(X(f)(x_1),y_2|A(f)(a_1))}{(p_{X_2}\circ\psi_2)(X(f)(x_1)|A(f)(a_1))}, & \text{if } (p_{X_2}\circ\psi_2)(X(f)(x_1)|A(f)(a_1)) > 0.\end{cases}
\end{aligned}
\tag{33}
$$

On the other hand, for all a_1 in A_1, x_1 in X_1, and y_2 in Y_2,

$$
\begin{aligned}
&Y(f)\circ(\psi|_X)_1\,(y_2|x_1,a_1)\\
&= \begin{cases} (Y(f)\circ p_{Y_1}\circ\psi_1)(y_1|a_1) & \text{if } p_{X_1}\circ\psi_1(X(f)^{-1}(X(f)(x_1))|a_1) = 0,\\ \sum_{y_1\in Y_1} Y(f)(y_2|y_1)\frac{\psi_1(X(f)^{-1}(X(f)(x_1)),y_1|a_1)}{(p_{X_1}\circ\psi_1)(X(f)^{-1}(X(f)(x_1))|a_1)} & \text{if } p_{X_1}\circ\psi_1(X(f)^{-1}(X(f)(x_1))|a_1) > 0\end{cases}\\
&= \begin{cases} (Y(f)\circ p_{Y_1}\circ\psi_1)(y_1|a_1) & \text{if } p_{X_1}\circ\psi_1(X(f)^{-1}(X(f)(x_1))|a_1) = 0,\\ \frac{\sum_{y_1\in Y_1} Y(f)(y_2|y_1)\psi_1(X(f)^{-1}(X(f)(x_1)),y_1|a_1)}{(p_{X_1}\circ\psi_1)(X(f)^{-1}(X(f)(x_1))|a_1)} & \text{if } p_{X_1}\circ\psi_1(X(f)^{-1}(X(f)(x_1))|a_1) > 0\end{cases}
\end{aligned}
\tag{34}
$$

We now have to prove that (33) and (34) are the same. We know that,

$$
\begin{tikzcd}
A_1 \arrow[r, "\psi_1"] \arrow[d, "A(f)"'] & X_1\times Y_1 \arrow[r, "p_{X_1}"] \arrow[d, "X(f)\otimes Y(f)"] & X_1 \arrow[d, "X(f)"] \\
A_2 \arrow[r, "\psi_2"'] & X_2\times Y_2 \arrow[r, "p_{X_2}"'] & X_2,
\end{tikzcd}
\tag{35}
$$

commutes, and since $A(f)$ and $X(f)$ are deterministic, for all a_1 in A_1 and x_1 in X_1,

$$
\begin{aligned}
p_{X_1}\circ\psi_1(X(f)^{-1}(X(f)(x_1))|a_1) &= (X(f)\circ p_{X_1}\circ\psi_1)(X(f)(x_1)|a_1)\\
=(p_{X_2}\circ\psi_2\circ A(f))(X(f)(x_1)|a_1) &= (p_{X_2}\circ\psi_2)(X(f)(x_1)|A(f)(a_1)).
\end{aligned}
$$

We also know that

$$
\begin{tikzcd}
A_1 \arrow[r, "\psi_1"] \arrow[d, "A(f)"'] & X_1\times Y_1 \arrow[r, "p_{X_1}"] \arrow[d, "X(f)\times Y(f)"] & Y_1 \arrow[d, "Y(f)"] \\
A_2 \arrow[r, "\psi_2"'] & X_2\times Y_2 \arrow[r, "p_{Y_2}"'] & Y_2,
\end{tikzcd}
$$

commutes, so that for all a_1 in A_1 and y_2 in Y_2,

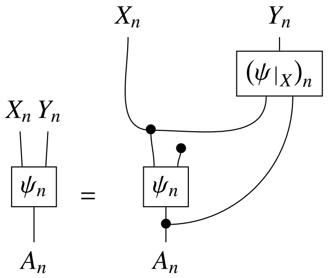
$$
(Y(f)\circ p_{Y_1}\circ\psi_1)(y_2|a_1) = (p_{Y_2}\circ\psi_2\circ A(f))(y_2|a_1) = (p_{Y_2}\circ\psi_2)(y_2|A(f)(a_1)).
$$

Lastly, we have,

$$\begin{aligned}
&\sum_{y_1\in Y_1} Y(f)(y_2|y_1)\psi_1(X(f)^{-1}(X(f)(x_1)), y_1|a_1)\\
=&((1_{X_2}\otimes Y(f))\circ(X(f)\otimes 1_{Y_1})\circ\psi_1)(X(f)(x_1), y_2|a_1)\\
=&((X(f)\otimes Y(f))\circ\psi_1)(X(f)(x_1), y_2|a_1)\\
=&(\psi_2\circ A(f))(X(f)(x_1), y_2|a_1)\\
=&\psi_2(X(f)(x_1), y_2|A(f)(a_1)),
\end{aligned}$$

where the first equality holds by definition (32), the second by functoriality of $-\otimes-$, the third by naturality of ψ (30) and the last one by determinism of $A(f)$. This yields that (33) and (34) are equal, and $\psi|_X$ is natural.

Now we would like $\psi|_X : X\times A\to Y$ to be a conditional, i.e., that



,

for $n=1,2$. In *FinStoch* this means by (28) that for all a_n in A_n, x_n in X_n and y_n in Y_n,

$$\psi_n(x_n, y_n|a_n) = (\psi|_X)_n(y_n|x_n, a_n)\cdot\sum_{y'_n\in Y_n}\psi_n(x_n, y'_n|a_n) = (\psi|_X)_n(y_n|x_n, a_n)\cdot(\psi_n)_{X_n}(x_n|a_n), \tag{36}$$

But since $(\psi|_X)_2 = \psi_2|_{X_2}$, we know that it holds for $n=2$. The case is more tedious for $n=1$. For all a_1 in A_1, x_1 in X_1, and y_1 in Y_1, let us see if the equality (36) holds.

- If $(p_{X_1}\circ\psi_1)(X(f)^{-1}(X(f)(x_1))|a_1)=0$, then $(p_{X_1}\circ\psi_1)(x_1|a_1)=0$ and $\psi_1(x_1,y_1|a_1)=0$, and the equality (36) holds.
- If $(p_{X_1}\circ\psi_1)(X(f)^{-1}(X(f)(x_1))|a_1)>0$, and if moreover $X(f)$ is injective, then

  $$\psi_1(y_1, X(f)^{-1}(X(f)(x_1))|a_1) = \psi_1(x_1, y_1|a_1),$$
  $$(p_{X_1}\circ\psi_1)(X(f)^{-1}(X(f)(x_1))|a_1) = p_{X_1}\circ\psi_1(x_1|a_1),$$

  and the equality (36) holds since

  $$\psi|_{X1}(y_1|x_1,a_1)\cdot(p_{X_1}\circ\psi_1(x_1|a_1)) = \frac{\psi_1(x_1,y_1|a_1)}{(p_{X_1}\circ\psi_1)(x_1|a_1)}\cdot(p_{X_1}\circ\psi_1(x_1|a_1)) = \psi_1(x_1,y_1|a_1).$$

  But if $X(f)$ is not injective, the equality fails. For example, if $A_1=A_2=\{*\}$ $X_1=\{1,2,3\}$, $X_2=\{0,1\}$ $Y_1=Y_2=\{a,b\}$, and

  $$\begin{aligned}
  A(f) =& 1_{\{*\}},\\
  X(f) :& 1\mapsto 1\\
  & 2\mapsto 1\\
  & 3\mapsto 0,\\
  Y(f) =& 1_{Y_1},\\
  \psi_1(1,a|*) =& \frac{1}{4}, \qquad \psi_1(1,b|*) = \frac{1}{12},\\
  \psi_1(2,a|*) =& \frac{1}{3}, \qquad \psi_1(2,b|*) = 0,\\
  \psi_1(3,a|*) =& \frac{1}{6}, \qquad \psi_1(3,b|*) = \frac{1}{6},\\
  \psi_2(0,a|*) =& \frac{1}{6}, \qquad \psi_2(0,b|*) = \frac{1}{6},\\
  \psi_2(1,a|*) =& \frac{7}{12}, \qquad \psi_2(1,b|*) = \frac{1}{12},
  \end{aligned}$$

then,

$$\begin{tikzcd} A_1 \arrow[r, "A(f)"] \arrow[d, "\psi_1"'] & A_2 \arrow[d, "\psi_2"] \\ X_1 \times Y_1 \arrow[r, "X(f)\otimes Y(f)"'] & X_2 \times Y_2, \end{tikzcd}$$

commutes but

$$\begin{aligned} \psi|_{X1}(a|1,*) \cdot (p_{X_1} \circ \psi_1)(1|*) &= \frac{\psi_1(1,a|*) + \psi_1(2,a|*)}{\psi_1(1,a|*) + \psi_1(1,b|*) + \psi_1(2,a|*) + \psi_1(2,b|*)} \\ &= \frac{\frac{1}{4}+\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} \cdot \frac{1}{3} = \frac{7}{24} \neq \frac{1}{4} = \psi_1(1,a|*). \end{aligned}$$

We don't know for now if there exists another way of defining conditionals on $Fun(\mathcal{D}, C_{det})$, but this example shows us that even for very basic and intuitive Markov categories, defining conditionals on the category of functor is not evident at all. It is also not evident to generalise the definition (31) to other Markov categories with conditionals, since the conditionals are not defined uniquely.

4 Generalisation of characteristics and theorems from statistical theory

One of the aim of categorical propbabilty is to generalise notions and Theorems from statistical theory to category theory. In this section, we will see how T. Fritz generalises the notions of sufficient statistics, almost surely properties, minimality, and Fisher-Neyman factorisation and Bahadur's Theorems.

4.1 Sufficient statistics

In statistics, a *statistical model* is an assumption on the model of the probability distribution for a sample space X. Formally, it is a pair $(\mathcal{P}, X)$ where X is the set of possible outcomes, and $\mathcal{P}$ is a set of possible probability distributions for X, indexed by a parameter θ, i.e., $\mathcal{P} = \{p_\theta : X \to [0,1] | \theta \in \Theta\}$. For example, the Gaussian model is a statistical model, whose parameters are the mean and the variance. T. Fritz generalises this to Markov categories as follows.

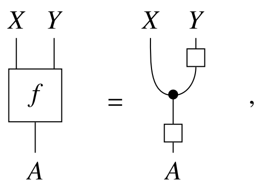
Definition 4.1. [2, 14.1.] Let C be a Markov category. Given an object $X \in C$, a *statistical model* with values in X consists of another object $\Theta \in C$ and a morphism $p : \Theta \to X$.

Remark. Sample spaces are generalised to objects in Markov categories. And a conditional probability density on a sample space X knowing a parameter θ in Θ, then the space of parameters is also generalised by an object Θ in a Markov category, and the conditional probability density is generalised by a morphism $\Theta \to X$. So statistical models ($\mathcal{P}$,X) are generalised by couples (p,X), with X an object $p : \Theta \to X$ a morphism in the Markov category.

Now a *statistic* for a statistical model is any value computed from the output of a probabilistic event, in order to estimate the parameter of the statistical model. For example, the mean is a statistic. In Markov categories, T. Fritz defines the notion of statistic as follows.

Definition 4.2. [2, 14.2.] Let C be a Markov category. A *statistic* for a statistical model $p : \Theta \to X$ in C is a deterministic morphism $s : X \to V$ to some other object V.

Definition 4.3. [2, 12.19.] Let C be a Markov category. Given a morphism $f : A \to X \otimes Y$ in C, we say that f *displays conditional independence* $A \perp\!\!\!\perp Y|X$ if it is of the form



,

where the blank boxes represent other, unnamed morphisms in C.

Example 4.4. In *FinStoch*, $f : A \to X \otimes Y$ displays conditional independence if there exists $g : A \to X$ and $h : X \to Y$ such that for all $x \in X, y \in Y, A \in A$

$$
\begin{aligned}
f(x,y|a) = &\sum_{x' \in X} (1_X \otimes h) \circ copy_X(x,y|x') \cdot g(x'|a) \\
= &\sum_{x' \in X} (1_X \otimes h)(x,y|x',x') \cdot g(x'|a) \\
= &\sum_{x' \in X} \delta_{x,x'} \cdot h(y|x') \cdot g(x'|a) = h(y|x)g(x|a).
\end{aligned}
$$

And we know that

$$f(x,y|a) = f_X(x|a) f|_X(y|x,a),$$

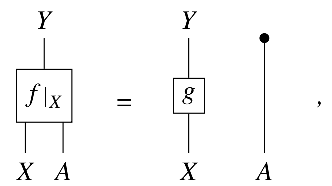
where f_X is the marginalisation of f on X. By marginalising on X, we can identify g with f_X,

$$
\begin{aligned}
f_X(x|a) = &\sum_{y \in Y} f(x,y|a) \\
= &\sum_{y \in Y} h(y|x)g(x|a) \\
= &g(x|a) \sum_{y \in Y} h(y|x) \\
= &g(x|a).
\end{aligned}
$$

So f displays the conditional independence if $h(y|x) = f|_X(y|x,a)$,i.e., if $f|_X(y|x,a)$ does not depend on a for all $y \in Y$ and $x \in X$, $a \in A$ such that $f_X(x|a) > 0$, which is the usual definition of conditional independance. We note that $f|_X(y|x,a)$ can depend on a where $f(x|a) = 0$, which means that the pair (x,a) has probability 0 to happen, so this makes sense that we can define arbitrarly the probability of y knowing this pair.

We can generalise this to any category C with conditionals.

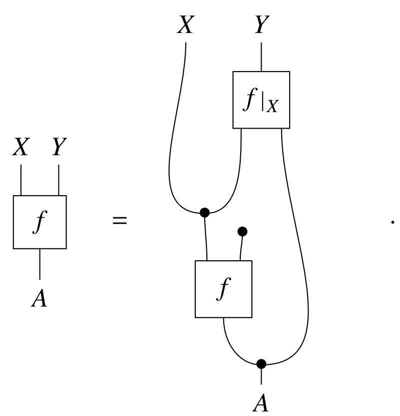
Proposition 4.5. *Let C be a category with conditionals. If morphism $f : A \to X \otimes Y$ in C is such that if $f|_X : X \otimes A \to Y$ does not depend on A, i.e., if there exists a morphism $g : X \to Y$ such that*



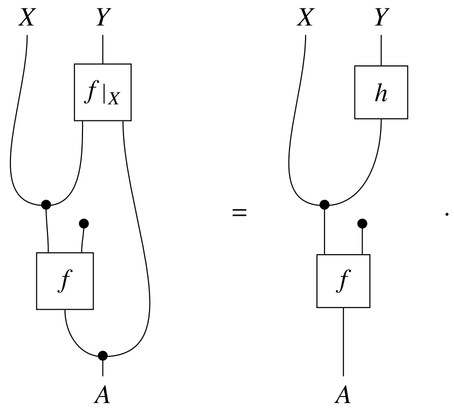
(37)

then f displays the conditional independence $A \perp\!\!\!\perp Y|X$.

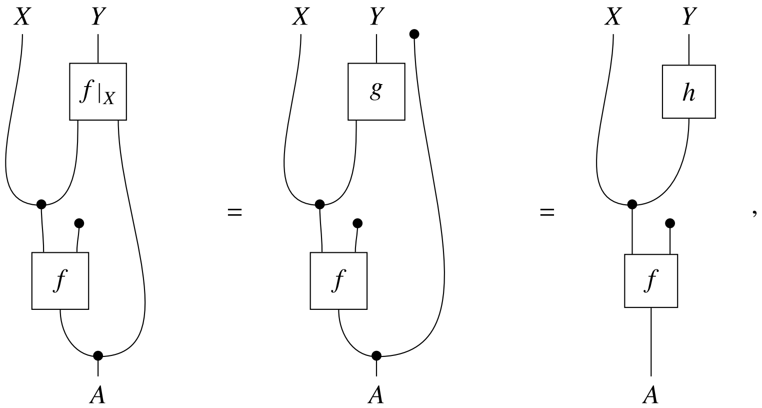
Proof. For all morphisms $f : A \to X \otimes Y$ in C,



It follows that f displays conditional independence $A \perp\!\!\!\perp Y|X$ if there exists $h : X \to Y$ such that

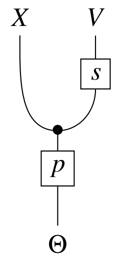


If $f|_X$ does not depend on A, we can define $h := g$ with g the morphism in (37) and



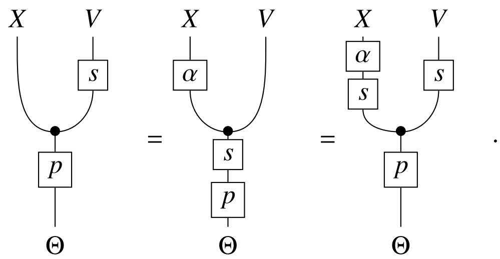
and f displays the conditional independence $A \perp\!\!\!\perp Y|X$. □

For $s : X \to V$ a statistic for a statistical model $p : \Theta \to X$, we can artificially construct a morphism displaying conditional independence $\Theta \perp\!\!\!\perp V|X$ by the joint distribution



In statistics, we are interested in *sufficient* statistics, i.e., statistics $s : X \to V$ on a model $p : \Theta \to X$ for which $s(X)$ does not depend on Θ, so that the value $s(X)$ is enough to estimate the parameter of the model $\theta \in \Theta$. T. Fritz gives the following definition for sufficient statistics in Markov categories.

Definition 4.6. A statistic $s : X \to V$ for a statistical model $p : \Theta \to X$ is *sufficient* if there is $\alpha : V \to X$ such that



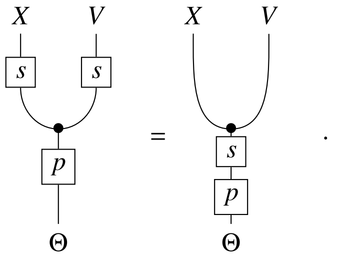
We interpret this definition as follows. A statistic $s : X \to V$ is sufficient if we can construct an *estimation* $e : V \to \Theta$ such that the following diagram commutes,

$$\begin{tikzcd}
\Theta \arrow[r, "p"] \arrow[dd, "p"'] & X \arrow[d, "s"] \\
 & V \arrow[d, "e"] \\
X & \Theta. \arrow[l, "p"]
\end{tikzcd}$$

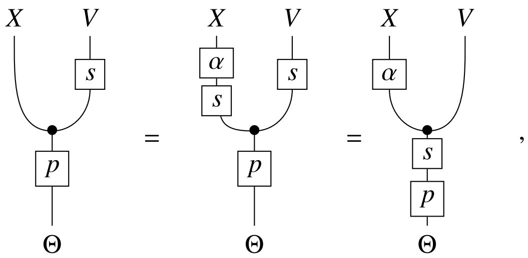
More generally, given the statistic $s(X) : X \to V$, we can find an estimation of the probability of X, $\alpha : V \to X$ such that the following diagram commutes,

$$\begin{tikzcd}
\Theta \arrow[r, "p"] \arrow[d, "p"'] & X \arrow[d, "s"] \\
X & V. \arrow[l, "\alpha"]
\end{tikzcd}$$

And since a statistic $s : X \to V$ is deterministic by definition,



We should then have,

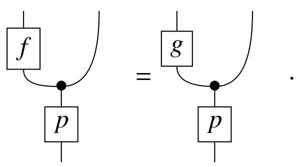


where the first equality holds if p is a particular morphism, called *complete*, which we will define in section 4.2. So this is not true in general but this gives an idea of what is a sufficient statistic.

4.2 Almost surely properties.

In statistics, we often speak about *almost surely* properties. If a property happens almost surely, it means that it happens with probability 1. In a Markov category, T. Fritz gives the following definitions for p-a.s. equality and deterministic morphisms.

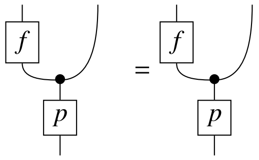
Definition 4.7. [2, 13.1.] Let C be a Markov category. Given any morphism $p : \Theta \to X$ in C, we say that $f, g : X \to Y$ in C are *p-a.s. equal* and write $f =_{p-a.s.} g$ if



Proposition 4.8. *Let C be a Markov category. Then for all morphisms $p : \Theta \to X$ in C the relation $=_{p-a.s.}$ is an equivalence relation. i.e., it satisfies*

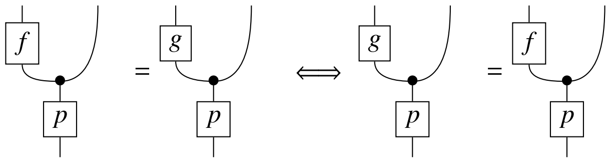
1. *reflexivity : for all morphism $f : X \to Y$ in C, $f =_{p-a.s.} f$;*
2. *symmetry : for all morphisms $f, g : X \to Y$ in C, $f =_{p-a.s.} g$ if and only if $g =_{p-a.s.} f$;*
3. *transitivity : for all morphisms $f, g, h : X \to Y$ in C, if $f =_{p-a.s.} g$ and $g =_{p-a.s.} h$, then $f =_{p-a.s.} h$.*

Proof. 1. We prove the reflexivity of $=_{p-a.s.}$. For all morphism $f : X \to Y$ in C, $f =_{p-a.s.} f$ since



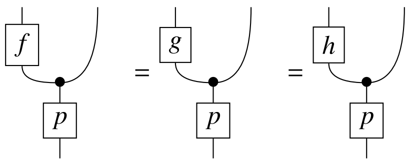
.

2. We prove symmetry of $=_{p-a.s.}$. For all morphisms $f, g : X \to Y$ in C, $f =_{p-a.s.} g$ if and only if $g =_{p-a.s.} f$ since



.

3. We prove transitivity of $=_{p-a.s.}$. For all morphisms $f, g, h : X \to Y$ in C, if $f =_{p-a.s.} g$ and $g =_{p-a.s.} h$, then $f =_{p-a.s.} h$ since



.

□

Example 4.9. In *FinStoch*, for $p : \Theta \to X$, $f, g : X \to Y$ are p-a.s. equal if, for all $\theta \in \Theta$ and $x \in X$, $y \in Y$,

$$\begin{aligned} & f(y|x)p(x|\theta) \\ & \qquad = g(y|x)p(x|\theta), \end{aligned} \tag{38}$$

i.e. if for all $\theta \in \Theta$, $f(y|x) = g(y|x)$ for all $y \in Y$, for all $x \in X$ such that $p(x|\theta) > 0$.

Definition 4.10. [2, 15.1.] Let C be a Markov category. A morphism $f : X \to Y$ in C is *complete* if for all $g, h : Y \to Z$ such that $gf = hf$,

$$g =_{f-a.s.} h.$$

In the theory of statistics, a statistic $s : X \to V$ for a model $p : \Theta \to X$ is *complete* if for all measurable function $g : V \to Y$ such that $g(s(X))$ has constant expectation with respect to the parameter θ, $g(s(X))$ equals its expectation with probability 1. It is *boundedly complete* if the condition above holds for all bounded measurable function $g : V \to Y$.

Example 4.11. Let us analyse the complete morphisms in *FinStoch*. Let $f : X \to Y$ a morphism in *FinStoch*. Here we will use the following notation.

Notation. We write

$$X = x_{i0 \leq i \leq n},$$

with n the cardinality of X. And similarly for Y,

$$Y = y_{i0 \leq i \leq m},$$

with m the cardinality of Y.

For all y in Y, we can define its probability $f(y|-)$,

$$f(y|-) := \begin{pmatrix} f(y|x_1) & f(y|x_2) & \dots & f(y|x_n) \end{pmatrix}, .$$

For all x in X, we define the image of x as follows.

$$f(-|x) := \begin{pmatrix} f(y_1|x) \\ f(y_2|x) \\ . \\ . \\ . \\ f(y_m|x) \end{pmatrix},$$

and we interpret it as a vector in the *probability simplex of* Y, *simp*(Y), defined as follows.

$$simp(Y) := \left\{ \sum_{i=1}^{m} \lambda_i \cdot e_i \mid 0 \leq \lambda_i \leq 1, \sum_{i=1}^{m} \lambda_i = 1 \right\},$$

with $e_i \in \mathbb{R}^m$ the basis vector with only 0s but a 1 at its ith position. And we call the set of linear combination of the images the *image of* f :

$$Im(f) := \left\{ \sum_{i=1}^{n} \lambda_i f(-|x_i) \mid \lambda_i \in \mathbb{R} \right\}.$$

Lastly, we call the following set *support* of f.

$$supp(f) := \{y|\exists x \in X \text{ s.t. } f(y|x) > 0\}.$$

As we have seen above, for all $g, h : Y \to Z$ in *FinStoch*, $g =_{f-a.s.} h$ if and only if they are equal on the support of f. On the other hand, if $gf = hf$, we have for all $x \in X$ and $z \in Z$,

$$\sum_{y \in Y} g(z|y) f(y|x) = \sum_{y \in Y} h(z|y) f(y|x)$$

i.e., in our notations,

$$g(z|-) \cdot f(-|x) = h(z|-) \cdot f(-|x),$$

and this happens if and only if for all $z \in Z$, $g(z|-) \cdot v = h(z|-) \cdot v$ for all $v \in Im(f)$. We now show that f is complete if and only if for all $y_i \in supp(f)$, $e_i \in (Im(f))$.

If for all $y_i \in supp(f)$, $e_i \in (Im(f))$, then if $gf = hf$, for all y_i in $Supp(f)$,

$$g(z|y_i) = g(z|-) \cdot e_i = h(z|-) \cdot e_i = f(z|y_i),$$

so $g = h$ on $supp(f)$, i.e. $g =_{p-a.s.} f$.

If there exists y_i in $supp(f)$ such that $e_i \notin Im(f)$, the idea is to find two applications $g(-|-)$ and $h(-|-)$ that are equal on $Im(f)$ but not on e_i. Let $\tilde{g} = Id_Y$ and $\tilde{h}(-|-)$ to be the projection matrix on $Im(f)$. Then these applications verify, for all v in $Im(f)$,

$$\tilde{g}(-|-) \cdot v = v = \tilde{f}(-|-) \cdot v,$$

so that

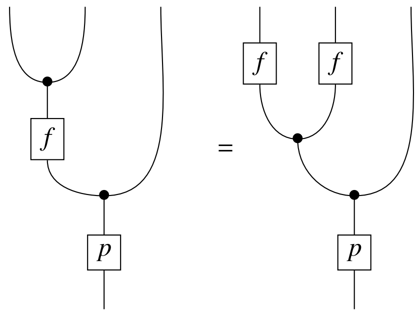
$$\tilde{g}(y|-) \cdot v = v = \tilde{f}(y|-) \cdot v,$$

for all y in Y, and $gf = hf$. But

$$\tilde{g}(-|y_i) = e_i \neq \tilde{f}(-|y_i),$$

since $\tilde{f}(-|y_i) \in Im(f)$. So there exists $y_j \in Y$ such that $\tilde{g}(y_j|y_i) = \delta_{i,j} \neq \tilde{f}(y_j|y_i)$, this means $\tilde{f}$ and $\tilde{g}$ are not equal on the support of f. But since $\tilde{h}(-|-)$ is not necessarly in the probability simplex of Y, it is not necessarly a morphism in *FinStoch*, so we need to apply a suitable invertible contraction and translation T to $\tilde{f}$ and $\tilde{g}$, and get $f := T\tilde{f}$ and $g := T\tilde{g}$ having the required properties.

Definition 4.12. [2, 13.11.] Given $p : \Theta \to X$, we say that $f : X \to Y$ is *p-a.s. deterministic* if



Example 4.13. In FinStoch, for $p : \Theta \to X$, a morphism $f : X \to Y$ is p-a.s. deterministic if for all x in X such that there exists $\theta \in \Theta$ such that $p(x|\theta) > 0$ (i.e., x is in $supp(p)$),

$$\delta_{y_1,y_2} \cdot f(y_1|x) = copy_Y \circ f(y_1, y_2|x) = (f \otimes f) \circ copy_X(y_1, y_2|x) = f(y_1|x)f(y_2|x)$$

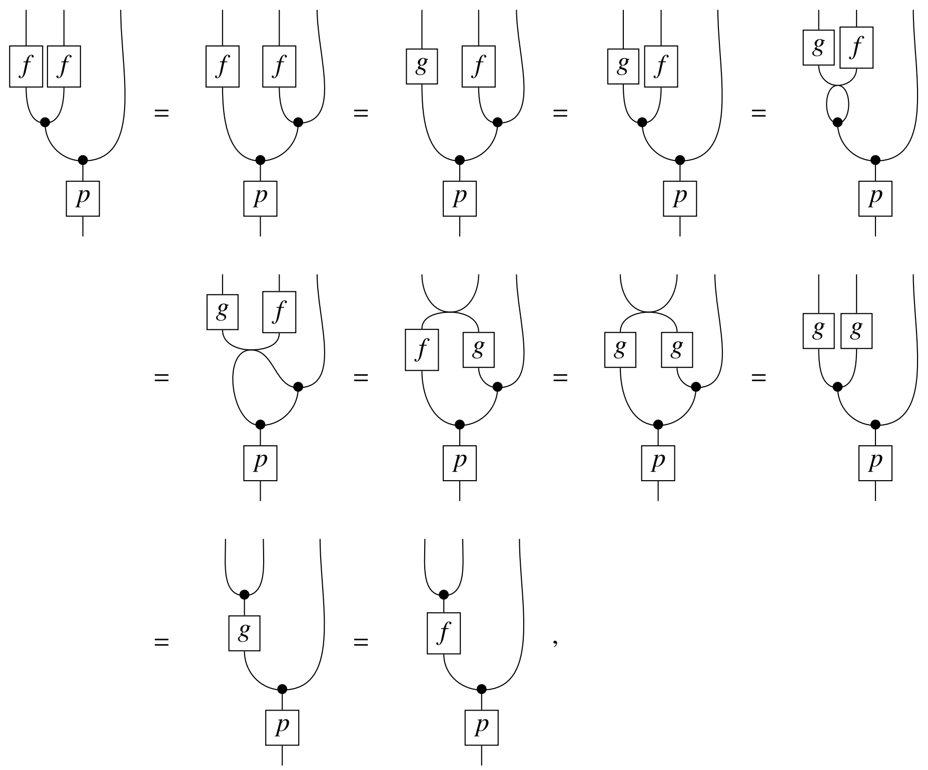
for all y_1, y_2 in Y using definition of p-a.s. equality in *FinStoch* (38). This is the same condition as for a morphism $f|_{supp(p)}$ to be deterministic in *FinStoch*. So there exists a function $\tilde{f} : supp(p) \to Y$ such that for all x in $supp(p)$,

$$f(y|x) = \delta_{y,\tilde{f}(x)},$$

for all y in Y.

Lemma 4.14. *[2, 13.12.] Let C be a Markov category, and $p : \Theta \to X$, $f, g : X \to Y$ be morphisms in C such that $g =_{p-a.s.} f$. If g is p-a.s. deterministic, then so is f.*

Proof.

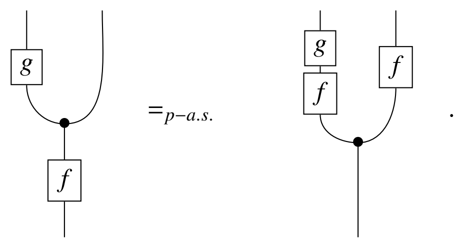


where we used coassociativity and cocommutativity of the comultiplication (11), naturality of the braiding and p-a.s. determinism of g. □

Definition 4.15. [2, 13.16.] Let C be a Markov category. We say that C is *strictly positive* if whenever morphisms

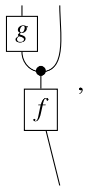
$$\Theta \xrightarrow{p} X \xrightarrow{f} Y \xrightarrow{g} Z$$

are such that gf is p-a.s. deterministic, then also

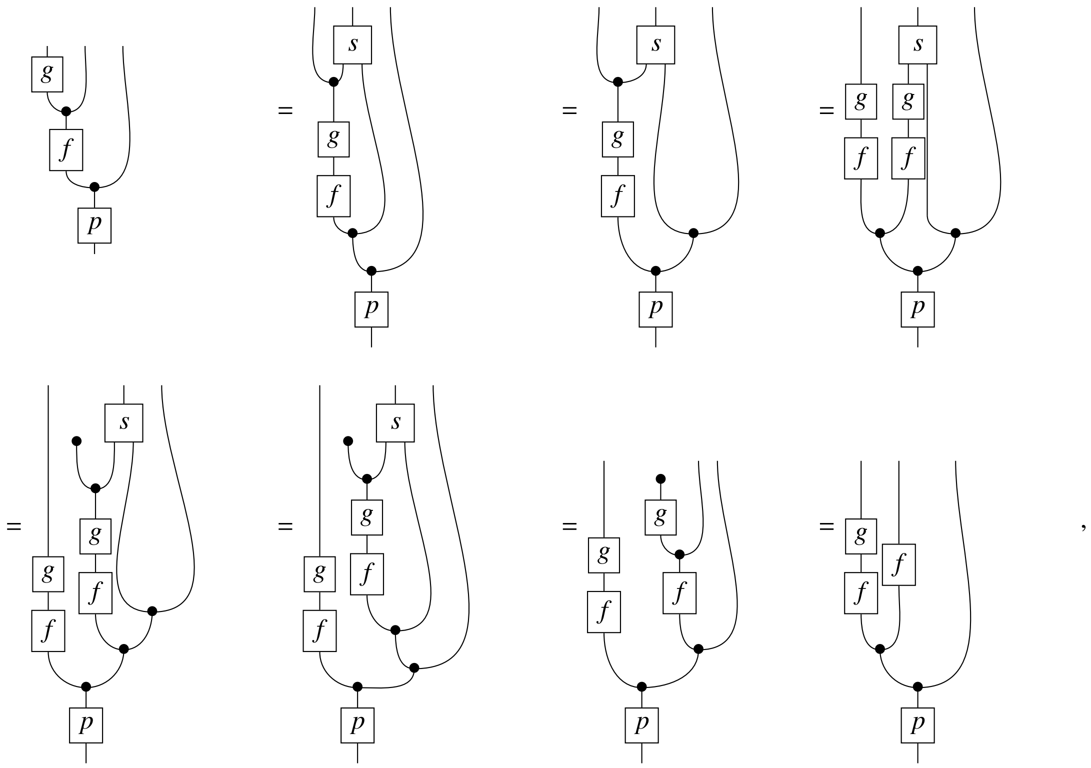


Proposition 4.16. *Let C be a Markov category. If C has conditionals, then C is strictly positive.*

Proof. Let the morphisms $p : \Theta \to X$, $f : X \to Y$ and $g : Y \to Z$ in C be such that gf is p-a.s. deterministic. Let $s : Z \otimes X \to Y$ be a conditional for the morphism



so that



where the first and sixth equality hold by defintion of s, the second fifth by (11), the third by p-a.s. determinism of gf and the fourth and last ones by (11) and (12). $\square$

Example 4.17. *FinStoch* is strictly positive.

Proposition 4.18. *[2, 13.17.] Let C be a Markov category, and $p : \Theta \to X$, $f : X \to Y$, $g : Y \to Z$ morphisms in C. Suppose that gf is p-a.s. deterministic, and that also*

1. *f is p-a.s. deterministic, or*
2. *C is strictly positive.*

Then g is fp-a.s. deterministic.

4.3 Fisher-Neyman Factorisation Theorem

In statistical theory, there exists a necessary and sufficient condition for a statistic to be sufficient given by the Fisher-Neyman Factorisation Theorem. In this section, we will see a condition in abstract theory. The Fisher-Neyman Factorisation Theorem says that a statistic $T : X \to V$ is sufficient for the statistical model $\mathcal{F} = \{f_\theta : X \to [0,1] \mid \theta \in \Theta\}$ if and only if there exists a statistical model $\mathcal{G} = \{g_\theta : X \to [0,1] \mid \theta \in \Theta\}$ and a nonnegative function $h : X \to [0,1]$ such that

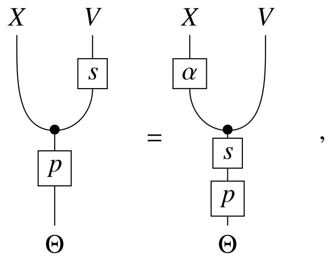
$$f_\theta(x) = h(x) g_\theta(T(x)).$$

T. Fritz gives a characterization for sufficient statistics in Markov categories, i.e. in terms of Definition 4.6.

Theorem 4.19. *[2, 14.5.] Let C be a strictly positive Markov category. Then a statistic $s : X \to V$ is sufficient for a statistical model $p : \Theta \to X$ if and only if there is $\alpha : V \to X$ such that $\alpha s p = p$ and $s\alpha =_{sp-a.s.} 1_V$.*

Proof.

- The only if direction holds since if $s : X \to V$ is sufficient for the statistical model $p : \Theta \to X$, there is $\alpha : V \to X$ such that



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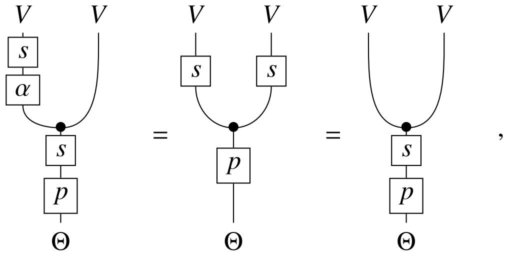
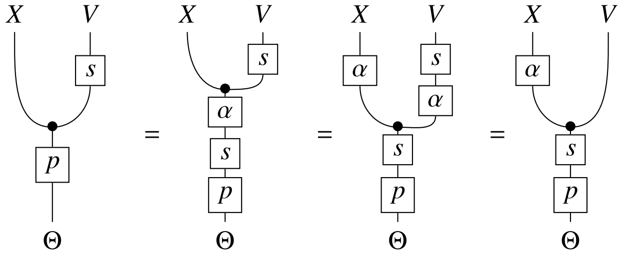
by marginalisation on X, we get that $\alpha s p = p$. Moreover, since s is deterministic, and using the defintion of α, there is $\alpha : V \to X$ such that



,

this proves $s\alpha =_{p-a.s.} 1_V$

- The "if" direction of the statement holds since if there is $\alpha : V \to X$ such that $\alpha s p = p$ and $s\alpha =_{sp-a.s.} 1_V$,



,

where the first equality holds since $\alpha s p = p$ and the last one since $s\alpha =_{sp-a.s.} 1_V$. For the second equality, we used that $s\alpha$ is deterministic by Lemma 4.14, and then since C is strictly positive, the equality holds. □

4.4 Minimality and Bahadur's theorem

The notion of minimality appears in statistics to compare the level of information a statistic can give. We will say that, for two statistics $s : X \to V$, $t : X \to W$ for the same model $p : \Theta \to X$, s is more informative than t if we can find t knowing s, i.e. if there is a function $c : W \to V$ such that cs gives the same information as t. There are several ways to define this notion in Markov categories. In [4, 7], P.V. Golubtsov defines minimality on IT-categories, which are the equivalent of Markov categories.

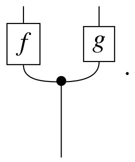
Definition 4.20. [4, 7.1] We will say that $\triangleright$ is an *accuracy relation* on an IT-category (Markov category) **C** if for any pair of objects A and B in C the set $C(A, B)$ of all ITs (morphisms) from A to B is equipped with a partial order $\triangleright$ that satisfies the following monotonicity conditions:

$$f \triangleright f', g \triangleright g' \implies g \circ f \triangleright g' \circ f',$$

for all morphisms $f, f' : A \to B$ and $g, g' : B \to C$, and

$$f \triangleright f', g \triangleright g' \implies f * g \triangleright f' * g'.$$

for all morphisms $f, f' : A \to B_1$, $g, g' : A \to B_2$, and where $f * g$ is the equivalent for us of



Definition 4.21. [4, 7.2] An information transformer (a morphism in a Markov category C) a is *more informative* (better) than b if there exists an information transformer (another morphism in C) c such that $c \circ a \triangleright b$, that is,

$$a \geqslant b \iff \exists c \quad c \circ a \triangleright b.$$

T. Fritz gives another definition, as follows.

Definition 4.22. [2, 16.1.] Let $s : X \to V$ and $t : X \to W$ be statistics for a statistical model $p : \Theta \to X$. Then we write $t \leq s$ if there is $c : V \to W$ such that $t =_{p-a.s.} cs$.

Remark. The defintion of Golubtsov is a generalisation of the one of T. Fritz if $=_{p-a.s.}$ is an accuracy relation, in terms of Definition 4.20, i.e., if

$$f =_{p-a.s.} f', g =_{p-a.s.} g' \Longrightarrow g \circ f =_{p-a.s.} g' \circ f',$$

for all morphisms $f, f' : A \to B$ and $g, g' : B \to C$ in C. The second condition holds in all Markov categories.

Proposition 4.23. *Let C be a Markov category. Then $\leq$ is a preorder for the set S_p of sufficient statistics for a given model $p : \Theta \to X$ in C, i.e., $\leq$ satisfies*

1. *reflexivity : For all s in S_p, $s \leq s$, and*
2. *transitivity : For all s, t, u in S_p, if $s \leq t$ and $t \leq u$, then $s \leq u$.*

Proof.
1. We prove the reflexivity of $\leq$. For all $s : X \to V$ in S_p, we can take $c = 1_V : V \to V$ and $s = cs$, so $s =_{p-a.s.} cs$ and $s \leq s$.
2. We prove the transitivity of $\leq$. For all $s : X \to U$, $t : X \to V$, $u : X \to W$ in S_p such that $s \leq t$ and $t \leq u$, we can find $c : V \to U$ and $d : W \to V$ such that $s =_{p-a.s.} ct$ and $t =_{p-a.s.} du$. Then we have that $cd : W \to U$ and $s =_{p-a.s.} cdu$, giving $s \leq u$.

□

Definition 4.24. Let $(X, \leq)$ be a preordered set. Then a *least element* x in X is an element of X such that, for all $x' \in X$, $x \leq x'$.

Definition 4.25. [2, 16.2.] A statistic $s : X \to T$ is *minimal sufficient* if it is a least element in the preordered set of sufficient statistics for the given model $p : \Theta \to X$.

Bahadur's Theorem tells us that in the case where there exists one minimal sufficient statistics, then a sufficient and boundedly complete statistic is also minimal sufficient. T. Fritz proves the same theorem for strictly positive Markov categories.

Theorem 4.26. *[2, 16.3.] Suppose that C is strictly positive. Let $p : \Theta \to X$ be a statistical model which has a minimal sufficient statistic. If $s : X \to V$ is a sufficient statistic for p such that sp is complete, then s is minimal sufficient.*

Proof. let t be a minimal sufficient statistic for $p : \Theta \to X$. Since s is a sufficient statistic for $p : \Theta \to X$, $t \leq s$. So there exists c such that $cs =_{p-a.s.} t$. We show that s is also minimal, by proving that $s \leq t$,i.e., we need to find d such that $s =_{p-a.s.} dt =_{p-a.s.} dcs$. Since s and $t = cs$ are sufficient and by Theorem 4.19, there exists α and β such that

$$\begin{aligned} \alpha sp = p, s\alpha =_{sp-a.s.} Id, \\ \beta csp = p, cs\beta =_{csp-a.s} Id, \end{aligned} \tag{39}$$

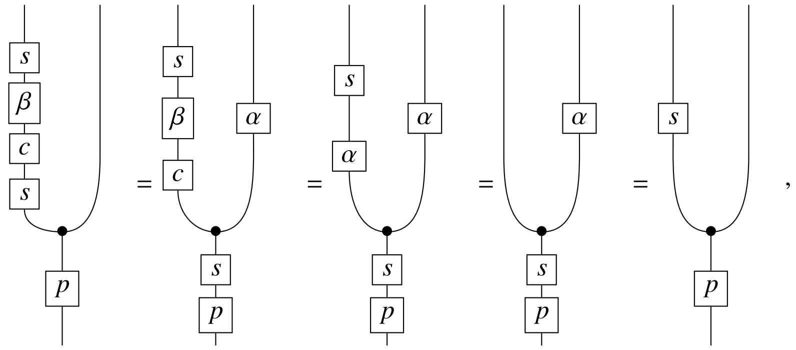
this implies that

$$\alpha sp = \beta csp,$$

and since sp is complete by assumption,

$$\alpha =_{sp-a.s.} \beta c. \tag{40}$$

Now we are ready to find d. Let us define $d := s\beta c$, so that



where the first and last identities hold by suffiencienty of s, the second by (40) and the third by the second identity in (39). □

5 Conclusion

This project allowed us to understand better the notion of Markov categories, and how it applies to standard probability theory. We have redifined the notions developed some of the arguments of Tobias Fritz in order to picture the state of the art in categorical probability theory. This is a very vast and exciting domain. Some questions that arised during the elaboration of this project remain unsolved, and it would be interesting to work further on it.

Problem 5.1. *Could we extend Lemma 2.20 to functors that happens to be symmetric monoidal, but not necessarly strict ?*

Problem 5.2. *Under what conditions the Eilenberg-Moore category of a monad T on a Markov category C can be endowed with a structure of Markov category ?*

Problem 5.3. *What properties of a Markov category C are preserved in the Kleisli category of a monad T satisfying conditions of Corollary 2.25 ?*

Problem 5.4. *Is it possible to generalise the results we have found in Example 3.18 to other starting categories than* Two *? Or to other Markov categories than* FinStoch *?*

Problem 5.5. *Let C be a Markov category such that there exists a small category $\mathcal{D}$ such that $Fun(\mathcal{D}, C_{det})$. Under what conditions does the constant functor $Cst : CtoFun(\mathcal{D}, C)$ admits a right or left adjoint, i.e. C is (co)complete ? Or inversely, does it helps that C is (co)complete for $Fun(\mathcal{D}, C_{det})$ to have conditionals ?*

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